

Finite Elements and Fast Iterative  
Solvers: with Applications in  
Incompressible Fluid Dynamics

---

Howard C. Elman     David J. Silvester  
Andrew J. Wathen

**OXFORD**  
UNIVERSITY PRESS

# CONTENTS

<b>0</b>	<b>Models of incompressible fluid flow</b>	1
<b>1</b>	<b>The Poisson equation</b>	10
1.1	Reference problems	11
1.2	Weak formulation	14
1.3	The Galerkin finite element method	17
1.3.1	Triangular finite elements ( $\mathbb{R}^2$ )	20
1.3.2	Quadrilateral elements ( $\mathbb{R}^2$ )	22
1.3.3	Tetrahedral elements ( $\mathbb{R}^3$ )	25
1.3.4	Brick elements ( $\mathbb{R}^3$ )	26
1.4	Implementation aspects	27
1.4.1	Triangular element matrices	28
1.4.2	Quadrilateral element matrices	31
1.4.3	Assembly of the Galerkin system	33
1.5	Theory of errors	36
1.5.1	A priori error bounds	38
1.5.2	A posteriori error bounds	48
1.6	Matrix properties	56
	Problems	61
	Computational exercises	66
<b>2</b>	<b>Solution of discrete Poisson problems</b>	68
2.1	The conjugate gradient method	69
2.1.1	Convergence analysis	73
2.1.2	Stopping criteria	75
2.2	Preconditioning	78
2.3	Singular systems are not a problem	83
2.4	The Lanczos and minimum residual methods	84
2.5	Multigrid	88
2.5.1	Two-grid convergence theory	95
2.5.2	Extending two-grid to multigrid	101
	Problems	107
	Computational exercises	110
<b>3</b>	<b>The convection–diffusion equation</b>	113
3.1	Reference problems	115
3.2	Weak formulation and the convection term	120
3.3	Approximation by finite elements	123
3.3.1	The Galerkin finite element method	123
3.3.2	The streamline diffusion method	126

3.4	Theory of errors	134
3.4.1	A priori error bounds	134
3.4.2	A posteriori error bounds	142
3.5	Matrix properties	148
3.5.1	Computational molecules and Fourier analysis	152
3.5.2	Analysis of difference equations	156
	Discussion and bibliographical notes	161
	Problems	163
	Computational exercises	164
<b>4</b>	<b>Solution of discrete convection–diffusion problems</b>	166
4.1	Krylov subspace methods	166
4.1.1	GMRES	167
4.1.2	Biorthogonalization methods	172
4.2	Preconditioning methods and splitting operators	176
4.2.1	Splitting operators for convection–diffusion systems	178
4.2.2	Matrix analysis of convergence	181
4.2.3	Asymptotic analysis of convergence	185
4.2.4	Practical considerations	190
4.3	Multigrid	194
4.3.1	Practical issues	195
4.3.2	Tools of analysis: smoothing and approximation properties	200
4.3.3	Smoothing	202
4.3.4	Analysis	205
	Discussion and bibliographical notes	208
	Problems	211
	Computational exercises	212
<b>5</b>	<b>The Stokes equations</b>	214
5.1	Reference problems	217
5.2	Weak formulation	222
5.3	Approximation using mixed finite elements	224
5.3.1	Stable rectangular elements ( $Q_2-Q_1, Q_2-P_{-1}, Q_2-P_0$ )	229
5.3.2	Stabilized rectangular elements ( $Q_1-P_0, Q_1-Q_1$ )	235
5.3.3	Triangular elements	245
5.3.4	Brick and tetrahedral elements	248
5.4	Theory of errors	249
5.4.1	A priori error bounds	250
5.4.2	A posteriori error bounds	262
5.5	Matrix properties	268
5.5.1	Stable mixed approximation	270
5.5.2	Stabilized mixed approximation	273
	Discussion and bibliographical notes	277
	Problems	280
	Computational exercises	283

<b>6</b>	<b>Solution of discrete Stokes problems</b>	285
6.1	The preconditioned MINRES method	286
6.2	Preconditioning	289
6.2.1	General strategies for preconditioning	291
6.2.2	Eigenvalue bounds	296
6.2.3	Equivalent norms for MINRES	303
6.2.4	MINRES convergence analysis	306
	Discussion and bibliographical notes	308
	Problems	309
	Computational exercises	310
<b>7</b>	<b>The Navier–Stokes equations</b>	313
7.1	Reference problems	315
7.2	Weak formulation and linearization	318
7.2.1	Stability theory and bifurcation analysis	320
7.2.2	Nonlinear iteration	324
7.3	Mixed finite element approximation	327
7.4	Theory of errors	330
7.4.1	A priori error bounds	331
7.4.2	A posteriori error bounds	333
	Discussion and bibliographical notes	337
	Problems	339
	Computational exercises	339
<b>8</b>	<b>Solution of discrete Navier–Stokes problems</b>	341
8.1	General strategies for preconditioning	342
8.2	Approximations to the Schur complement operator	346
8.2.1	The pressure convection-diffusion preconditioner	347
8.2.2	The least-squares commutator preconditioner	353
8.3	Performance and analysis	354
8.3.1	Ideal versions of the preconditioners	355
8.3.2	Use of iterative methods for subproblems	359
8.3.3	Convergence analysis	364
8.3.4	Enclosed flow: singular systems are not a problem	365
8.3.5	Relation to SIMPLE iteration	368
8.4	Nonlinear iteration	370
	Discussion and bibliographical notes	375
	Problems	378
	Computational exercises	379
	<b>Bibliography</b>	382
	<b>Index</b>	397