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Yu. V. Egorov M. A. Shubin

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Linear Partial Differential Equations. Foundations of the Classical Theory

Yu. V. Egorov, M. A. Shubin

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Preface

This volume contains a general introduction to the classical theory of linear partial differential equations for nonspecialist mathematicians and physicists.

Examples of partial differential equations are found as early as the papers of Newton and Leibniz, but the systematic study of them was begun by Euler. From the time of Euler on the theory of partial differential equations has occupied a central place in analysis, mainly because of its direct connections with physics and other natural sciences, as well as with geometry. In this connection the theory of linear equations has undergone a very profound and diverse development.

The present volume is introductory to a series of volumes devoted to the theory of linear partial differential equations. We could not encompass all aspects of the classical theory, and we did not try to do so. In writing this volume we did not hesitate to repeat ourselves in those situations where it seemed to us that repetition would facilitate the reading. However we have attempted to give a sketch of all the ideas that seemed fundamental to us, making no claim to completeness, of course. The reader who wishes to form a deeper acquaintance with some aspect of the theory discussed here may turn to the following, more specialized volumes in this series. In particular, many of the ideas of the modern theory are described in the authors' article published in the next volume.

The bibliography of this volume also makes no claim to completeness. We have attempted to cite as far as possible only textbooks, monographs, and survey articles.

The authors thank B. R. Vajnberg, who wrote Sect. 2.7, and M. S. Agranovich, who read this volume in manuscript and made many valuable remarks that enabled us to improve the exposition.