Integral, Probability, and Fractal Measures

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With 36 Figures

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Preface

This book may be considered a continuation of my Springer-Verlag text *Measure, Topology, and Fractal Geometry*. It presupposes some elementary knowledge of fractal geometry and the mathematics behind fractal geometry. Such knowledge might be obtained by study of *Measure, Topology, and Fractal Geometry* or by study of one of the other mathematically oriented texts (such as [13] or [87]). I hope this book will be appropriate to mathematics students at the beginning graduate level in the U.S. Most references are numbered and may be found at the end of the book; but *Measure, Topology, and Fractal Geometry* is referred to as [*MTFG*].

One of the reviews of [*MTFG*] says that it "sacrific[es] breadth of coverage for systematic development"[1]—although I did not have it so clearly formulated as that in my mind at the time I was writing the book, I think that remark is exactly on target. That sacrifice has been made in this volume as well. In many cases, I do not include the most general or most complete form of a result. Sometimes I have only an example of an important development. The goal was to omit most material that is too tedious or that requires too much background.

In this volume, the reader will again learn some of the mathematical background to be used in our study of fractal topics. Chapter 2 deals with integration in the modern sense. Since [*MTFG*] dealt with measures and had no theory of integrals, we have here used the knowledge of measures to aid in the discussion of integrals. Chapter 4 deals with mathematical probability. A mathematician may sometimes be inclined to think of probability as a branch of measure theory, but some of its motivations and techniques are quite different from the ones commonly seen in measure theory. In both of these chapters, only parts of the complete theory (of integrals, or of probability) are included—emphasis is on those particular results that are used elsewhere in the book for our discussion of fractals.

What is a **fractal**? There is, as yet, no widely accepted definition.

In [176, p. 15], Benoit Mandelbrot writes: *A fractal is by definition a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension.* In the notation of [*MTFG*], a "fractal" is a set E with $\operatorname{ind} E < \dim E$. I will sometimes call this a **fractal in the sense of Mandelbrot**. But note that in the second printing of 1982 (and more explicitly in [178, p. 8])

[1] Alec Norton's review in the *American Mathematical Monthly*, April 1992.

Mandelbrot states that he now prefers *to leave the term "fractal" without a pedantic definition* [176, p. 548].

S. James Taylor [255] has proposed that a fractal is a set for which the Hausdorff dimension coincides with the packing dimension. That is, a "fractal" is a set E with $\dim E = \operatorname{Dim} E$. I will sometimes call this a **fractal in the sense of Taylor**. But in fact, Taylor also is no longer promoting his definition.

Thus we are left with no precise definition of the term "fractal"; so of course there can be no theorems about "fractals" as such. Our theorems will be stated about sets E with certain properties, appropriate for the purpose we have in mind.

The term **fractal measure** in the title of the book has three possible meanings. All three will be seen in the course of the book:

(1) A "fractal measure" could be one of the measures (like $\mathcal{H}^s$ or $\mathcal{P}^s$) associated with the measurement of the s-dimensional "size" of a set and thus associated with the definition of various fractal dimensions (like the Hausdorff dimension or the packing dimension). We begin with such fractal measures in Chapter 1.

(2) M. Barnsley [13, §IX.5] suggests that a "fractal" really is an element of the space $\mathfrak{P}(S)$ of probability measures on a metric space S. Just as elements of $\mathfrak{K}(S)$ are "fractal sets," so elements of $\mathfrak{P}(S)$ are "fractal measures." Dimensions may be associated with measures in many of the same ways that they are associated with sets. Such fractal measures, in particular "self-similar" fractal measures, are found in Chapter 3.

(3) B. Mandelbrot [81] has proposed the term "fractal measure" for a kind of decomposition arising when a natural measure on a fractal set is not completely uniform: rather than a single number (the fractal dimension), the set may be decomposed into parts, each exhibiting its own dimension. Another term used for this decomposition is "multifractal." An example of this is in §5.7.

When I was first planning to write this book, I intended to include more material on random fractals; in particular, more material on the dimensions of sets associated with Brownian motion. But during the time that this book was under preparation there appeared a fine book by P. Mattila [186]. It contains much more along those lines than I could hope to include. So in this book I have contented myself with a brief introduction to the possibilities (Chapter 5). Mattila's book is more demanding on the reader in terms of the required background, but in return the results are much more complete.

In a similar way, there appeared a book by P. Massopust [185]; it contains material on fractal functions that will not be duplicated here, except for a few special cases in Chapter 3.

Here are some remarks on notation. Usually, we follow the notation of [*MTFG*]. But because of my use of uppercase script letters for measures, I have replaced some of my former uses of script letters by other notation. We will use Fraktur (German) letters to represent certain spaces: $\mathfrak{K}(S)$ the space of nonempty compact subsets of a metric space S with Hausdorff metric [*MTFG*,

p. 66]; $\mathfrak{C}(S,T)$ the space of continuous functions from S to T with uniform metric (when S is compact) [*MTFG*, p. 61]; $\mathfrak{P}(S)$ the space of probability measures on S with Hutchinson metric. We write $\mathbb{R}$ for the set of real numbers, $\mathbb{Q}$ for the set of rational numbers, $\mathbb{Z}$ for the set of integers, and $\mathbb{N}$ for the set $\{1, 2, 3, \cdots\}$ of natural numbers. In a metric space, diam A is the diameter of a set A, $B_r(x)$ is an open ball, $\overline{B}_r(x)$ is a closed ball, ∂A is the boundary of a set A, and $\overline{A}$ is the closure of a set A. The maximum of two real numbers a, b is written $a \vee b$, and the minimum is written $a \wedge b$. If a sequence x_n increases to a limit x, we write $x_n \nearrow x$. In $\mathbb{R}^d$ we write $|x|$ for the Euclidean norm of x, and $\mathcal{L}^d$ for d-dimensional Lebesgue measure. The Dirac measure at a point a is the measure $\mathcal{E}_a$ defined such that $\mathcal{E}_a(A) = 1$ if $a \in A$ but $\mathcal{E}_a(A) = 0$ if $a \notin A$. The indicator function of a set A is the function $\mathbb{1}_A$ defined by $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ if $x \notin A$. (Thus $\mathbb{1}_A(a) = \mathcal{E}_a(A)$.)

A metric space S is **totally bounded** iff for every $n \in \mathbb{N}$ and every $\varepsilon > 0$ there is a finite set A such that every point of S is within distance ε of a point of A. (A metric space is compact if and only if it is complete and totally bounded. This was essentially proved in passing in [*MTFG*].)

There are two different uses of the Greek letter ω in this book—I hope they will not be confused. One use is to represent a sample point in probability theory (Chapter 4). The other is to represent the least infinite ordinal. That is the meaning in the superscript $E^{(\omega)}$. It indicates that the letters in the strings $\sigma \in E^{(\omega)}$ are to be understood as ordered in that way (a first letter, a second letter, and so on).

Thanks are due to Edmund Mullins and Jeffrey Golds, who read parts of the text and found many errors.

Exercises will be found throughout the text. The reader is invited to provide proofs or investigate a topic further. Some of them are easy, some of them are hard, and a few I do not know how to solve. They should always be understood in the sense of "prove or disprove"; when an assertion turns out to be wrong, try to see what you can do to salvage it. Some of the exercises have hints (or even solutions) elsewhere in the text.

The ninety-nine figures in the text were drawn by the author on a Power Macintosh using these programs: Maple V, Photoshop, MacDraw Pro, LogoMation, and Fractal Attraction. The text was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX, with XY-pic and some macros provided by Springer-Verlag.

Columbus, Ohio — Gerald A. Edgar

Textbooks and Heaven only are Ideal.
—John Updike, *Scientific American* (January 1969)

Contents

* Asterisks indicate optional sections.