

Graduate Texts in Mathematics **123**

Readings in Mathematics

Editorial Board

S. Axler F. W. Gehring P. R. Halmos

Springer Science+Business Media, LLC

Graduate Texts in Mathematics

Readings in Mathematics

Ebbinghaus/Hermes/Hirzebruch/Koecher/Mainzer/Neukirch/Prestel/Remmert: *Numbers*

Fulton/Harris: *Representation Theory: A First Course*

Remmert: *Theory of Complex Functions*

Undergraduate Texts in Mathematics

Readings in Mathematics

Anglin: *Mathematics: A Concise History and Philosophy*

Anglin/Lambek: *The Heritage of Thales*

Bressoud: *Second Year Calculus*

Hairer/Wanner: *Analysis by Its History*

Hämmerlin/Hoffmann: *Numerical Mathematics*

Isaac: *The Pleasures of Probability*

Samuel: *Projective Geometry*

H.-D. Ebbinghaus H. Hermes
F. Hirzebruch M. Koecher K. Mainzer
J. Neukirch A. Prestel R. Remmert

Numbers

With an Introduction by K. Lamotke
Translated by H.L.S. Orde
Edited by J.H. Ewing

With 24 Illustrations



Springer

Heinz-Dieter Ebbinghaus
Hans Hermes
Mathematisches Institut
Universität Freiburg
Albertstraße 23b, D-79104
Freiburg, Germany

Friedrich Hirzebruch
Max-Planck-Institut für
Mathematik
Gottfried-Claren-Straße 26
D-53225 Bonn, Germany

Klaus Lamotke (*Editor of
German Edition*)
Mathematisches Institut
der Universität zu Köln
Weyertal 86–90, D50931
Köln, Germany

Max Koecher (1924–1990)
Reinhold Remmert
Mathematisches Institut
Universität Münster
Einsteinstraße 62
D-48149 Münster, Germany

Klaus Mainzer
Lehrstuhl für Philosophie und
Wissenschaftstheorie
Universität Augsburg
Universitätsstraße 10
D-86195 Augsburg, Germany

H.L.S. Orde (*Translator*)
Bressenden
Biddenden near Ashford
Kent TN27 8DU, UK

Jürgen Neukirch
Fachbereich Mathematik
Universitätsstraße 31
D-93053 Regensburg, Germany

Alexander Prestel
Fakultät für Mathematik
Universität Konstanz
Postfach 5560, D-78434
Konstanz, Germany

John H. Ewing (*Editor of
English Edition*)
Department of Mathematics
Indiana University
Bloomington, IN 47405, USA

Editorial Board

S. Axler
Department of Mathematics
Michigan State University
East Lansing, MI 48824, USA

F.W. Gehring
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109, USA

P.R. Halmos
Department of Mathematics
Santa Clara University
Santa Clara, CA 95053, USA

Mathematics Subject Classification (1991): 11-XX, 11-03

Library of Congress Cataloging-in-Publication Data

Zahlen, *Grundwissen Mathematik* 1. English
Numbers / Heinz-Dieter Ebbinghaus . . . [et al.]; with an
introduction by Klaus Lamotke; translated by H.L.S. Orde; edited
by John H. Ewing.

p. cm.—(Readings in mathematics)

Includes bibliographical references.

ISBN 978-0-387-97497-2 ISBN 978-1-4612-1005-4 (eBook)

DOI 10.1007/978-1-4612-1005-4

1. Number theory. I. Ebbinghaus, Heinz-Dieter. II. Ewing,

John H. III. Series: Graduate texts in mathematics. Readings
in mathematics.

QA241.Z3413 1991

512'.7—dc20

89-48588

Printed on acid-free paper.

This book is a translation of the second edition of *Zahlen, Grundwissen Mathematik 1*, Springer-Verlag, 1988. The present volume is the first softcover edition of the previously published hardcover version ISBN 978-0-387-97497-2

© 1991 Springer Science+Business Media New York

Originally published by Springer-Verlag New York Inc. in 1991

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer Science+Business Media, LLC), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden. The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

Camera-ready copy prepared using LaTeX.

9 8 7 6 5 4 3 (Corrected third printing, 1995)

ISBN 978-0-387-97497-2

Preface to the English Edition

A book about numbers sounds rather dull. This one is not. Instead it is a lively story about one thread of mathematics—the concept of “number”—told by eight authors and organized into a historical narrative that leads the reader from ancient Egypt to the late twentieth century. It is a story that begins with some of the simplest ideas of mathematics and ends with some of the most complex. It is a story that mathematicians, both amateur and professional, ought to know.

Why write about numbers? Mathematicians have always found it difficult to develop broad perspective about their subject. While we each view our specialty as having roots in the past, and sometimes having connections to other specialties in the present, we seldom see the panorama of mathematical development over thousands of years. *Numbers* attempts to give that broad perspective, from hieroglyphs to K -theory, from Dedekind cuts to nonstandard analysis. Who first used the standard notation for π (and who made it standard)? Who were the “quaternionists” (and can their zeal for quaternions tell us anything about the recent controversy concerning Chaos)? What happened to the endless supply of “hypercomplex numbers” or to quaternionic function theory? How can the study of maps from projective space to itself give information about algebras? How did mathematicians resurrect the “ghosts of departed quantities” by reintroducing infinitesimals after 200 years? How can games be numbers and numbers be games? This is mathematical culture, but it’s not the sort of culture one finds in scholarly tomes; it’s lively culture, meant to entertain as well as to inform.

This is not a book for the faint-hearted, however. While it starts with material that every undergraduate could (and should) learn, the reader is progressively challenged as the chapters progress into the twentieth century. The chapters often tell about people and events, but they primarily tell about mathematics. Undergraduates can certainly read large parts of this book, but mastering the material in late chapters requires work, even for mature mathematicians. This is a book that can be read on several levels, by amateurs and professionals alike.

The German edition of this book, *Zahlen*, has been quite successful. There was a temptation to abbreviate the English language translation by making it less complete and more compact. We have instead tried to produce a faithful translation of the entire original, which can serve as a scholarly reference as well as casual reading. For this reason, quotations

are included along with translations and references to source material in foreign languages are included along with additional references (usually more recent) in English.

Translations seldom come into the world without some labor pains. Authors and translators never agree completely, especially when there are eight authors and one translator, all of whom speak both languages. My job was to act as referee in questions of language and style, and I did so in a way that likely made neither side happy. I apologize to all.

Finally, I would like to thank my colleague, Max Zorn, for his helpful advice about terminology, especially his insistence on the word “octonions” rather than “octaves.”

March 1990

John Ewing

Preface to Second Edition

The welcome which has been given to this book on numbers has pleasantly surprised the authors and the editor. The scepticism which some of us had felt about its concept has been dispelled by the reactions of students, colleagues and reviewers. We are therefore very glad to bring out a second edition—much sooner than had been expected. We have willingly taken up the suggestion of readers to include an additional chapter by J. NEUKIRCH on p -adic numbers. The chapter containing the theorems of FROBENIUS and HOPF has been enlarged to include the GELFAND–MAZUR theorem. We have also carefully revised all the other chapters and made some improvements in many places. In doing so we have been able to take account of many helpful comments made by readers for which we take this opportunity of thanking them. P. ULLRICH of Münster who had already prepared the name and subject indexes for the first edition has again helped us with the preparation of the second edition and deserves our thanks.

Oberwolfach, March 1988

Authors and Publisher

Preface to First Edition

The *basic mathematical knowledge* acquired by every mathematician in the course of his studies develops into a unified whole only through an awareness of the multiplicity of relationships between the individual mathematical theories. Interrelationships between the different mathematical disciplines often reveal themselves by studying historical development. One of the main underlying aims of this series is to make the reader aware that mathematics does not consist of isolated theories, developed side by side, but should be looked upon as an organic whole.

The present book on numbers represents a departure from the other volumes of the series inasmuch as seven authors and an editor have together contributed thirteen chapters. In conversations with one another the authors agreed on their contributions, and the editor endeavored to bring them into harmony by reading the contributions with a critical eye and holding subsequent discussions with the authors. The other volumes of the series can be studied independently of this one.

While it is impossible to name here all those who have helped us by their comments, we should nevertheless like to mention particularly Herr Gericke (of Freiburg) who helped us on many occasions to present the historical development in its true perspective.

K. Peters (at that time with Springer-Verlag) played a vital part in arranging the first meeting between the publisher and the authors. The meetings were made possible by the financial support of the Volkswagen Foundation and Springer-Verlag, as well as by the hospitality of the Mathematical Research Institute in Oberwolfach.

To all of these we extend our gratitude.

Oberwolfach, July 1983

Authors and Editor

Contents

<i>Preface to the English Edition</i>	v
<i>Preface to Second Edition</i>	vii
<i>Preface to First Edition</i>	ix
<i>Introduction, K. Lamotke</i>	1
Part A. From the Natural Numbers, to the Complex Numbers, to the p-adics	7
<i>Chapter 1. Natural Numbers, Integers, and Rational Numbers.</i> K. Mainzer	9
§1. Historical	9
1. Egyptians and Babylonians. 2. Greece. 3. Indo-Arabic Arithmetical Practice. 4. Modern Times	
§2. Natural Numbers	14
1. Definition of the Natural Numbers. 2. The Recursion Theorem and the Uniqueness of \mathbb{N} . 3. Addition, Multiplication and Ordering of the Natural Numbers. 4. PEANO'S AXIOMS	
§3. The Integers	19
1. The Additive Group \mathbb{Z} . 2. The Integral Domain \mathbb{Z} . 3. The Order Relation in \mathbb{Z}	
§4. The Rational Numbers	22
1. Historical. 2. The Field \mathbb{Q} . 3. The Ordering of \mathbb{Q}	
References	23
<i>Chapter 2. Real Numbers.</i> K. Mainzer	27
§1. Historical	27
1. HIPPASUS and the Pentagon. 2. EUDOXUS and the Theory of Proportion. 3. Irrational Numbers in Modern Mathematics. 4. The Formulation of More Precise Definitions in the Nineteenth Century	
§2. DEDEKIND Cuts	36
1. The Set \mathbb{R} of Cuts. 2. The Order Relation in \mathbb{R} .	

3. Addition in \mathbb{R} . 4. Multiplication in \mathbb{R}	
§3. Fundamental Sequences	39
1. Historical Remarks. 2. CAUCHY's Criterion for Convergence. 3. The Ring of Fundamental Sequences. 4. The Residue Class Field F/N of Fundamental Sequences Modulo the Null Sequence. 5. The Completely Ordered Residue Class Field F/N	
§4. Nesting of Intervals	43
1. Historical Remarks. 2. Nested Intervals and Completeness	
§5. Axiomatic Definition of Real Numbers	46
1. The Natural Numbers, the Integers, and the Rational Numbers in the Real Number Field. 2. Completeness Theorem. 3. Existence and Uniqueness of the Real Numbers	
References	51
<i>Chapter 3. Complex Numbers.</i> R. Remmert	55
§1. Genesis of the Complex Numbers	56
1. CARDANO (1501–1576). 2. BOMBELLI (1526–1572). 3. DESCARTES (1596–1650), NEWTON (1643–1727) and LEIBNIZ (1646–1716). 4. EULER (1707–1783). 5. WALLIS (1616–1703), WESSEL (1745–1818) and ARGAND (1768–1822). 6. GAUSS (1777–1855). 7. CAUCHY (1789–1857). 8. HAMILTON (1805–1865). 9. Later Developments	
§2. The Field \mathbb{C}	65
1. Definition by Pairs of Real Numbers. 2. The Imaginary Unit i . 3. Geometric Representation. 4. Impossibility of Ordering the Field \mathbb{C} . 5. Representation by Means of 2×2 Real Matrices	
§3. Algebraic Properties of the Field \mathbb{C}	71
1. The Conjugation $\mathbb{C} \rightarrow \mathbb{C}$, $z \mapsto \bar{z}$. 2. The Field Automorphisms of \mathbb{C} . 3. The Natural Scalar Product $\operatorname{Re}(w\bar{z})$ and Euclidean Length $ z $. 4. Product Rule and the "Two Squares" Theorem. 5. Quadratic Roots and Quadratic Equations. 6. Square Roots and n th Roots	
§4. Geometric Properties of the Field \mathbb{C}	78
1. The Identity $\langle w, z \rangle^2 + \langle iw, z \rangle^2 = w ^2 z ^2$. 2. Cosine Theorem and the Triangle Inequality. 3. Numbers on Straight Lines and Circles. Cross-Ratio. 4. Cyclic Quadrilaterals and Cross-Ratio. 5. PTOLEMY's Theorem. 6. WALLACE's Line.	
§5. The Groups $O(\mathbb{C})$ and $SO(2)$	85
1. Distance Preserving Mappings of \mathbb{C} . 2. The Group $O(\mathbb{C})$. 3. The Group $SO(2)$ and the Isomorphism $S^1 \rightarrow SO(2)$.	

4. Rational Parametrization of Properly Orthogonal 2×2 Matrices.	
§6. Polar Coordinates and n th Roots	89
1. Polar Coordinates. 2. Multiplication of Complex Numbers in Polar Coordinates. 3. DE MOIVRE's Formula. 4. Roots in Unity.	
<i>Chapter 4. The Fundamental Theorem of Algebra.</i>	
R. Remmert	97
§1. On the History of the Fundamental Theorem	98
1. GIRARD (1595–1632) and DESCARTES (1596–1650). 2. LEIBNIZ (1646–1716). 3. EULER (1707–1783). 4. D'ALEMBERT (1717–1783). 5. LAGRANGE (1736–1813) and LAPLACE (1749–1827). 6. GAUSS's Critique. 7. GAUSS's Four Proofs. 8. ARGAND (1768–1822) and CAUCHY (1798–1857). 9. The Fundamental Theorem of Algebra: Then and Now. 10. Brief Biographical Notes on Carl Friedrich GAUSS	
§2. Proof of the Fundamental Theorem Based on ARGAND	111
1. CAUCHY's Minimum Theorem. 2. Proof of the Fundamental Theorem. 3. Proof of ARGAND's Inequality. 4. Variant of the Proof. 5. Constructive Proofs of the Fundamental Theorem.	
§3. Application of the Fundamental Theorem	115
1. Factorization Lemma. 2. Factorization of Complex Polynomials. 3. Factorization of Real Polynomials. 4. Existence of Eigenvalues. 5. Prime Polynomials in $\mathbb{C}[Z]$ and $\mathbb{R}[X]$. 6. Uniqueness of \mathbb{C} . 7. The Prospects for "Hypercomplex Numbers."	
Appendix. Proof of the Fundamental Theorem, after LAPLACE	120
1. Results Used. 2. Proof. 3. Historical Note	
<i>Chapter 5. What is π?</i> R. Remmert	123
§1. On the History of π	124
1. Definition by Measuring a Circle. 2. Practical Approximations. 3. Systematic Approximation. 4. Analytical Formulae. 5. BALTZER's Definition. 6. LANDAU and His Contemporary Critics	
§2. The Exponential Homomorphism $\exp: \mathbb{C} \rightarrow \mathbb{C}^\times$	131
1. The Addition Theorem. 2. Elementary Consequences. 3. Epimorphism Theorem. 4. The Kernel of the Exponential Homomorphism. Definition of π . Appendix. Elementary Proof of Lemma 3.	
§3. Classical Characterizations of π	137
1. Definitions of $\cos z$ and $\sin z$. 2. Addition Theorem.	

3. The Number π and the Zeros of $\cos z$ and $\sin z$. 4. The Number π and the Periods of $\exp z$, $\cos z$ and $\sin z$. 5. The Inequality $\sin y > 0$ for $0 < y < \pi$ and the Equation $e^{i\frac{\pi}{2}} = i$. 6. The Polar Coordinate Epimorphism $p: \mathbb{R} \rightarrow S^1$. 7. The Number π and the Circumference and Area of a Circle.	
§4. Classical Formulae for π	142
1. LEIBNIZ's Series for π . 2. VIETA's Product Formula for π . 3. EULER's Product for the Sine and WALLIS's Product for π . 4. EULER's Series for π^2, π^4, \dots 5. The WEIERSTRASS Definition of π . 6. The Irrationality of π and Its Continued Fraction Expansion. 7. Transcendence of π .	
<i>Chapter 6. The p-Adic Numbers.</i> J. Neukirch	155
§1. Numbers as Functions	155
§2. The Arithmetic Significance of the p -Adic Numbers	162
§3. The Analytical Nature of p -Adic Numbers	166
§4. The p -Adic Numbers	173
References	177
Part B. Real Division Algebras	179
<i>Introduction,</i> M. Koecher, R. Remmert	181
<i>Repertory. Basic Concepts from the Theory of Algebras,</i> M. Koecher, R. Remmert	183
1. Real Algebras. 2. Examples of Real Algebras. 3. Subalgebras and Algebra Homomorphisms. 4. Determination of All One-Dimensional Algebras. 5. Division Algebras. 6. Construction of Algebras by Means of Bases	
<i>Chapter 7. Hamilton's Quaternions.</i> M. Koecher, R. Remmert	189
Introduction	189
§1. The Quaternion Algebra \mathbb{H}	194
1. The Algebra \mathbb{H} of the Quaternions. 2. The Matrix Algebra \mathcal{H} and the Isomorphism $F: \mathbb{H} \rightarrow \mathcal{H}$. 3. The Imaginary Space of \mathbb{H} . 4. Quaternion Product, Vector Product and Scalar Product. 5. Noncommutativity of \mathbb{H} . The Center. 6. The Endomorphisms of the \mathbb{R} -Vector Space \mathbb{H} . 7. Quaternion Multiplication and Vector Analysis. 8. The Fundamental Theorem of Algebra for Quaternions.	
§2. The Algebra \mathbb{H} as a Euclidean Vector Space	206
1. Conjugation and the Linear Form $\Re e$. 2. Properties of	

the Scalar Product. 3. The “Four Squares Theorem”. 4. Preservation of Length, and of the Conjugacy Relation Under Automorphisms. 5. The Group S^3 of Quaternions of Length 1. 6. The Special Unitary Group $SU(2)$ and the Isomorphism $S^3 \rightarrow SU(2)$.	
§3. The Orthogonal Groups $O(3)$, $O(4)$ and Quaternions 1. Orthogonal Groups. 2. The Group $O(\mathbb{H})$. CAYLEY’S Theorem. 3. The Group $O(\text{Im } \mathbb{H})$. HAMILTON’S Theorem. 4. The Epimorphisms $S^3 \rightarrow SO(3)$ and $S^3 \times S^3 \rightarrow SO(4)$. 5. Axis of Rotation and Angle of Rotation. 6. EULER’S Parametric Representation of $SO(3)$.	213
<i>Chapter 8. The Isomorphism Theorems of FROBENIUS, HOPF and GELFAND–MAZUR.</i> M. Koecher, R. Remmert	221
Introduction	221
§1. Hamiltonian Triples in Alternative Algebras 1. The Purely Imaginary Elements of an Algebra. 2. Hamiltonian Triple. 3. Existence of Hamiltonian Triples in Alternative Algebras. 4. Alternative Algebras.	223
§2. FROBENIUS’S Theorem 1. FROBENIUS’S Lemma. 2. Examples of Quadratic Algebras. 3. Quaternions Lemma. 4. Theorem of FROBENIUS (1877)	227
§3. HOPF’S Theorem 1. Topologization of Real Algebras. 2. The Quadratic Mapping $\mathcal{A} \rightarrow \mathcal{A}$, $x \mapsto x^2$. HOPF’S Lemma. 3. HOPF’S Theorem. 4. The Original Proof by HOPF. 5. Description of All 2-Dimensional Algebras with Unit Element	230
§4. The GELFAND–MAZUR Theorem 1. BANACH Algebras. 2. The Binomial Series. 3. Local Inversion Theorem. 4. The Multiplicative Group \mathcal{A}^\times . 5. The GELFAND–MAZUR Theorem. 6. Structure of Normed Associative Division Algebras. 7. The Spectrum. 8. Historical Remarks on the GELFAND–MAZUR Theorem. 9. Further Developments	238
<i>Chapter 9. CAYLEY Numbers or Alternative Division Algebras.</i> M. Koecher, R. Remmert	249
§1. Alternative Quadratic Algebras 1. Quadratic Algebras. 2. Theorem on the Bilinear Form. 3. Theorem on the Conjugation Mapping. 4. The Triple Product Identity. 5. The Euclidean Vector Space \mathcal{A} and the Orthogonal Group $O(\mathcal{A})$	250
§2. Existence and Properties of Octonions 1. Construction of the Quadratic Algebra \mathbb{O} of Octonions.	256

2. The Imaginary Space, Linear Form, Bilinear Form, and Conjugation of \mathbb{O} . 3. \mathbb{O} as an Alternative Division Algebra. 4. The “Eight-Squares” Theorem. 5. The Equation $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}p$. 6. Multiplication Table for \mathbb{O}	
§3. Uniqueness of the CAYLEY Algebra	261
1. Duplication Theorem. 2. Uniqueness of the CAYLEY Algebra (Zorn 1933). 3. Description of \mathbb{O} by ZORN’s Vector Matrices	
<i>Chapter 10. Composition Algebras. HURWITZ’s Theorem— Vector-Product Algebras. M. Koecher, R. Remmert</i>	265
§1. Composition Algebras	267
1. Historical Remarks on the Theory of Composition. 2. Examples. 3. Composition Algebras with Unit Element. 4. Structure Theorem for Composition Algebras with Unit Element	
§2. Mutation of Composition Algebras	272
1. Mutation of Algebras. 2. Mutation Theorem for Finite-Dimensional Composition Algebras. 3. HURWITZ’s Theorem (1898)	
§3. Vector-Product Algebras	275
1. The Concept of a Vector-Product Algebra. 2. Construction of Vector-Product Algebras. 3. Specification of all Vector-Product Algebras. 4. MALCEV-Algebras. 5. Historical Remarks	
<i>Chapter 11. Division Algebras and Topology.</i> F. Hirzebruch	281
§1. The Dimension of a Division Algebra Is a Power of 2	281
1. Odd Mappings and HOPF’s Theorem. 2. Homology and Cohomology with Coefficients in F_2 . 3. Proof of HOPF’s Theorem. 4. Historical Remarks on Homology and Cohomology Theory. 5. STIEFEL’s Characteristic Homology Classes	
§2. The Dimension of a Division Algebra Is 1, 2, 4 or 8	290
1. The mod 2 Invariants $\alpha(f)$. 2. Parallelizability of Spheres and Division Algebras. 3. Vector Bundles. 4. WHITNEY’s Characteristic Cohomology Classes. 5. The Ring of Vector Bundles. 6. Bott Periodicity. 7. Characteristic Classes of Direct Sums and Tensor Products. 8. End of the Proof. 9. Historical Remarks	
§3. Additional Remarks	299
1. Definition of the HOPF Invariant. 2. The HOPF Construction. 3. ADAMS’s Theorem on the HOPF Invariants. 4. Summary. 5. ADAMS’s Theorem About Vector Fields on Spheres	

Contents	xvii
References	301
Part C. Infinitesimals, Games, and Sets	303
<i>Chapter 12. Nonstandard Analysis.</i> A. Prestel	305
§1. Introduction	305
§2. The Nonstandard Number Domain ${}^*\mathbb{R}$	309
1. Construction of ${}^*\mathbb{R}$. 2. Properties of ${}^*\mathbb{R}$	
§3. Features Common to \mathbb{R} and ${}^*\mathbb{R}$	316
§4. Differential and Integral Calculus	321
1. Differentiation. 2. Integration	
Epilogue	326
References	327
<i>Chapter 13. Numbers and Games.</i> H. Hermes	329
§1. Introduction	329
1. The Traditional Construction of the Real Numbers.	
2. The CONWAY Method. 3. Synopsis	
§2. CONWAY Games	331
1. Discussion of the DEDEKIND Postulates. 2. CONWAY's Modification of the DEDEKIND Postulates. 3. CONWAY Games	
§3. Games	334
1. The Concept of a Game. 2. Examples of Games. 3. An Induction Principle for Games	
§4. On the Theory of Games	336
1. Winning Strategies. 2. Positive and Negative Games. 3. A Classification of Games	
§5. A Partially Ordered Group of Equivalent Games	339
1. The Negative of a Game. 2. The Sum of Two Games. 3. Isomorphic Games. 4. A Partial Ordering of Games. 5. Equality of Games	
§6. Games and CONWAY Games	343
1. The Fundamental Mappings. 2. Extending to CONWAY Games the Definitions of the Relations and Operations Defined for Games. 3. Examples	
§7. CONWAY Numbers	346
1. The CONWAY Postulates (C1) and (C2). 2. Elementary Properties of the Order Relation. 3. Examples	
§8. The Field of CONWAY Numbers	349
1. The Arithmetic Operations for Numbers. 2. Examples. 3. Properties of the Field of Numbers	
References	353

<i>Chapter 14. Set Theory and Mathematics.</i>	
H.-D. Ebbinghaus	355
Introduction	355
§1. Sets and Mathematical Objects	358
1. Individuals and More Complex Objects. 2. Set Theoretical Definitions of More Complex Objects.	
3. Urelements as Sets	
§2. Axiom Systems of Set Theory	363
1. The RUSSELL Antinomy. 2. ZERMELO's and the ZERMELO- FRAENKEL Set Theory. 3. Some Consequences. 4. Set Theory with Classes	
§3. Some Metamathematical Aspects	372
1. The VON NEUMANN Hierarchy. 2. The Axiom of Choice. 3. Independence Proofs	
Epilogue	378
References	378
<i>Name Index</i>	381
<i>Subject Index</i>	387
<i>Portraits of Famous Mathematicians</i>	393