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G. Duvaut J. L. Lions

Inequalities in Mechanics and Physics

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Georges Duvaut
Université de Paris VI, Mécanique Théoretique, 75230 Paris, France

Jacques Louis Lions
Collège de France, 75015 Paris, France

Translator: Charlotte W. John
New Rochelle, New York 10804, U.S.A.

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Introduction

1. We begin by giving a simple example of a *partial differential inequality* that occurs in an elementary physics problem.

We consider a fluid with pressure $u(x,t)$ at the point x at the instant t that occupies a region Ω of $\mathbb{R}^3$ bounded by a membrane Γ of negligible thickness that, however, is *semi-permeable*, i.e., a membrane that permits the fluid to enter Ω freely but that prevents all outflow of fluid.

One can prove then (cf. the details in Chapter I, Section 2.2.1) that

$$\frac{\partial u}{\partial t} - \Delta u = g \left(\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \right) \quad \text{in} \quad \Omega, t>0, \tag{1}$$

g a given function, *with boundary conditions in the form of inequalities*[1]

$$\begin{aligned} &u(x,t)>0 \Rightarrow \partial u(x,t)/\partial n=0, \quad x\in\Gamma, \\ &u(x,t)=0 \Rightarrow \partial u(x,t)/\partial n\geqslant 0, \quad x\in\Gamma, \end{aligned} \tag{2}$$

to which is added the initial condition

$$u(x,0)=u_0(x). \tag{3}$$

We note that conditions (2) are *non linear*; they imply that, at each fixed instant t, there exist on Γ two regions Γ_0^t and Γ_1^t where $u(x,t)=0$ and $\partial u(x,t)/\partial n=0$, respectively. These regions are not prescribed; thus we deal with a "free boundary" problem.

We can restate (1), (2) in the (equivalent) form of *inequalities*. For that purpose, we introduce the set K of "test functions" v:

$$K = \{v| \quad v=\text{function defined in } \Omega\ ^2,\ v\geqslant 0 \text{ on } \Gamma\}; \tag{4}$$

then (1), (2) are *equivalent* to

$$\begin{aligned} &u(.,t)\in K \quad \forall t\geqslant 0, \\ &\int_\Omega \left[\frac{\partial u}{\partial t}(v-u)+\text{grad}_x u.\text{grad}_x(v-u)-g(v-u)\right] \mathrm{d}x\geqslant 0 \quad \forall v\in K. \end{aligned} \tag{5}$$

[1] $\partial/\partial n$ denotes the derivative in the direction of the normal to Γ directed towards the exterior of Ω.

[2] We must take v in the Sobolev space $H^1(\Omega)$; this will beformulated more precisely in Chapter 1.

The problem to find a solution u of (5) with the initial condition (3) is what we call *an inequality of evolution* (of parabolic type).

2. The preceding example has features of a general character: we will encounter problems *that can be expressed in terms of inequalities* in situations where the constraints, the equations of state, the physical laws change when certain thresholds are crossed or attained.

The aim of the present work is to discuss examples of such situations in Mechanics and Physics.

3. The "program" indicated in the above paragraph covers an immense field that we have not studied exhaustively; we limited ourselves in this volume to the simplest classic laws. We have treated the following subjects:

1) problems of semi-permeable walls, of diffusion, applications to thermodynamics and hydrodynamics;
2) problems of control, particularly in theromodynamics;
3) problems in (linearized) elasticity involving friction and unilateral conditions;
4) problems of bending of plane plates;
5) phenomena of elastic-visco-plasticity, perfect elasticity, plasticity, rigid-visco-plasticity, rigid-perfect plasticity, and locking materials;
6) flows of Bingham fluids;
7) problems of inequalities connected with the system of Maxwell operators.

4. In order to avoid ambiguity in the formulation of the problems enumerated above, it was necessary to give a concise but precise review of the mechanical or physical bases for the situations envisioned. This is done at the beginning of each chapter. We now give a short description of the contents of the chapters.

5. In Section 1. above, we gave an example of the problems treated in *Chapter* 1; other problems concern temperature control.

In *Chapter* 2, control problems are discussed that lead to inequalities of the type (compare with (5))

$$\partial u/\partial t(.,t)\in K,$$

$$(6)\qquad \int_\Omega \left[\frac{\partial u}{\partial t}\left(v-\frac{\partial u}{\partial t}\right)+\operatorname{grad}_x u.\operatorname{grad}_x\left(v-\frac{\partial u}{\partial t}\right)-g\left(v-\frac{\partial u}{\partial t}\right)\right]\mathrm{d}x\geqslant 0 \qquad \forall v\in K,$$

with the initial condition (3).

Chapter 3 treats the classic linear theory of elasticity rather completely (in particular, we give a proof of Korn's inequality, the indispensable mathematical basis for the theory); then we go on to problems of friction that lead to inequalities; we adopt Coulomb's law and indicate some modifications.

Chapter 4 deals with problems of friction connected with the mechanics of thin plates.

Chapter 5 is devoted to phenomena of elasto-visco plasticity from which we derive, by various passages to the limit, the elastic-perfectly plastic case, the rigid-visco-plastic case and the rigid-perfectly plastic case, all of these problems being stated in the form of inequalities. In this chapter, we also investigate Hencky's law and locking materials.

Chapter 6 treats the flow of a certain type of non-newtonian fluid: Bingham fluids. Here, we are led to inqualities of evolution containing, as a special case, the classic system of Navier-Stokes equations.

Chapter 7 is concerned with the problems of inequalities connected with the system of Maxwell operators. We will first study conducting media where the relation between the electric field and the current density is expressed by the classic Ohm's law, i. e., media with constant resistivity (we call such a medium "stable"). Subsequently, we treat the case of media susceptible to ionization under the influence of the electric field. The resistivity then abruptly becomes infinite: these are the phenomena that occur in connection with breakdown of condensors or antennas.

"Hybrid" problems simultaneously involving two of the situations described in the outline for the preceding chapters are treated in separate articles by the authors (see: Duvaut-Lions [7], [8]).

6. Throughout this book, we made use of the most direct methods possible, generally representing *inequalities* (the absolutely indispensable tool, especially for *problems of evolution*) as *limiting cases of non linear equations* (which, moreover, usually have a mechanical or physical interpretation).

In addition, in order to facilitate the reading of the book, we presented each chapter as independent as possible (at the price of some repetition).

7. There are numerous earlier works on stationary inequalities in Mechanics. The classic approach (see P. Germain [1], G. Mandel [1], E. Tonti [1] and the bibliographies of these works) consists in studying stationary elasticity in relation to minimization of quadratic functionals on *vector spaces*. The minimization of analogous functionals on convex sets that *are not* vector spaces made its appearance in *perfect plasticity* (where the stress tensor remains in a closed *bounded* convex set) (cf. W.I. Koiter [1], G. Mandel [2], W. Prager [1] and the bibliographies of these works), subsequently in *unilateral elasticity* in the problem of Signorini, solved in G. Fichera [1], then in J.L. Lions-G. Stampacchia [1].

Similarly, the phenomena of cavitation studied by J. Moreau [3] and the investigation of minimal surfaces with constraints (J.C. Nitsche [1]) also lead to problems in variational inequalities.

The *inequalities of evolution* were introduced in Lions-Stampacchia for the parabolic case, in Lions [4] for the hyperbolic case and have been investigated particularly by H. Brézis [2][3] (cf. also the book Lions [1] and the bibliography of this work). It seems that the applications of the inequalities of evolution to Mechanics and Physics are being investigated here for the first time. As might be expected, these applications lead to many new problems, some of them still open; we mention specifically:

- the problem of *regularity* of solutions (the methods of Brézis-Stampacchia [1], Brézis [2] are not applicable to numerous situations in this book);
- the problem of inequalities of evolution in connection with convex sets or with functions *depending on* t (they occur particularly in the theory of dynamic elastic-visco plasticity.

[3] where one will find, in particular, the use of the theory of *non linear semigroups*, a theory that has not been used in this book.

8. There are other situations in physics leading to inequalities, either stationary or of evolution. We will return to this subject, e.g. in the discussion of thermo-elastic-visco plasticity and of optimal control of systems governed by inequalities. We also would like to point out that a free boundary problem, occurring in hydrodynamics, was solved with inequality methods by C. Baiocchi [1].

We did not treat two subjects related to this book:

i) *singular perturbations* related to inequalities (theory of singular layers); we refer to J.L. Lions [5], [6];

ii) methods of *numerical approximation* of solutions of inequalities of evolution, methods that will be treated in the book by R. Glowinski, J.L. Lions and R. Trémolières [1]. We refer to the works on this subject by D. Bégis [1], J.F. Bourgat [1], H. Brézis et M. Sibony [1], J. Céa et R. Glowinski [1], J. Céa, R. Glowinski et J. Nédelec [1], R. Commincioli [1], [2], [3], B. Courjaret [1], M. Frémond [1], A. Fusciardi, U. Mosco, F. Scarpini et A. Schiaffino [1], M. Goursat [1], Y. Haugazeau [1], P.G. Hodge [1], A. Marrocco [1], M. Sibony [1], D. Viaud [1].

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G. Duvaut, J.L. Lions