

Dale R. Durran

# Numerical Methods for Wave Equations in Geophysical Fluid Dynamics

With 93 Illustrations



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