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J. L. Doob

Classical Potential Theory and Its Probabilistic Counterpart

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Introduction

Potential theory and certain aspects of probability theory are intimately related, perhaps most obviously in that the transition function determining a Markov process can be used to define the Green function of a potential theory. Thus it is possible to define and develop many potential theoretic concepts probabilistically, a procedure potential theorists observe with jaundiced eyes in view of the fact that now as in the past their subject provides the motivation for much of Markov process theory. However that may be it is clear that certain concepts in potential theory correspond closely to concepts in probability theory, specifically to concepts in martingale theory. For example, superharmonic functions correspond to supermartingales. More specifically: the Fatou type boundary limit theorems in potential theory correspond to supermartingale convergence theorems; the limit properties of monotone sequences of superharmonic functions correspond surprisingly closely to limit properties of monotone sequences of supermartingales; certain positive superharmonic functions [supermartingales] are called "potentials," have associated measures in their respective theories and are subject to domination principles (inequalities) involving the supports of those measures; in each theory there is a reduction operation whose properties are the same in the two theories and these reductions induce sweeping (balayage) of the measures associated with potentials, and so on.

The purpose of this book is to develop this correspondence between potential theory and probability theory by examining in detail classical potential theory, that is, the potential theory of Laplace's equation, together with the corresponding probability theory, that is, martingale theory. The joining link which makes this correspondence especially perspicuous is the Brownian motion process, so this process is studied as needed. In order to carry through this program it is necessary to study parabolic potential theory, that is, the potential theory of the heat equation, and the corresponding process of space time Brownian motion. No knowledge of potential theory is presupposed but it is assumed that the reader is familiar with basic probability concepts through conditional expectations. The necessary lattice theory, analytic set theory and capacity theory are covered in the Appendixes.

Thus this book on the one hand contains an introduction to classical and parabolic potential theory and on the other hand contains an introduction to martingale theory, including a smattering of the general theory of stochastic processes and of Markov process theory. There is cross referencing between the nonprobabilistic and probabilistic aspects of the work, and the linking of classical and parabolic potential theory with martingale theory, by Brownian motion and space time Brownian motion, is examined in depth.

One natural criticism of this project is that there is no reason to treat the very special potential theories of the Laplace and heat equations rather than general axiomatic potential theory. Another criticism is that there is no reason to treat potential theory other than as a special subhead of Markov process theory. In the author's opinion, however, classical potential theory is too important to serve merely as a source of illustrations of axiomatic potential theory, which theory in turn is too important in its own right to be left to the probabilists. To learn potential theory from probability is like learning algebraic geometry without the geometry.

It would be quite impossible to cover all those parts of modernized classical potential theory which are relevant to the purpose of this book. Thus there are striking gaps. For example the treatment of energy is skimpy, and Dirichlet spaces and the concept of bounded mean oscillation are not even mentioned in the text. The emphasis is on the Dirichlet problem and related topics; these are treated in considerable depth. The treatments of classical and parabolic potential theories are sometimes separated, sometimes together, but the notation is designed to exhibit the parallelism of the two theories: dots in the notation distinguish parabolic from classical concepts, thereby muddling eyes but saving brains. And the martingale theory notation is designed to point out to readers the corresponding potential theory notation.

Only the part of Markov process theory needed for the relevant discussion of Brownian motion and conditional Brownian motion is covered. In this book a stochastic process is a specified family of random variables, frequently coupled with a filtration to which the family is adapted, but the measure space of the process is left unspecified and there is no translation operator. Thus in a discussion of Brownian motion from a varying initial point the measure space on which the process is defined may vary with the initial point. This definition of a process may not be best for general Markov process theory but is convenient in the special context of this book; it implies for example that no matter how or on what measure space a process is defined, if it has the properties of a Brownian motion (continuous sample functions and the correct distributions of independent increments) then it is a Brownian motion. In a traditional song, a child finds an object which looks smells and tastes like a peanut so the child concludes that the object is a peanut. As stochastic processes are sometimes defined, with special properties demanded of the measure space on which the process random variables are defined, this simple logic is invalid. However the point of view of this book makes it essential in discussing Brownian motion to prove certain invariance properties, for example that two Brownian motion processes in N space, with a common initial point and variance parameter, have the same probability of hitting an analytic set. This fact is not trivial and such questions are treated.

There is nothing very novel in this book. Potential theorists may find the treatment of reductions on boundary sets of interest, as well as the use of iterated reductions to obtain limit theorems. Correspondingly, probabilists may find the new supermartingale crossing inequalities and the technique of iterated reductions of supermartingales of interest. A new domination principle for supermartingales illustrates the fact that classical potential theory still suggests interesting probability results.

The author thanks Bruce Hajek, Naresh Jain and John Taylor for helpful comments on various chapters and, finally, thanks his typist: usually faithful, sometimes accurate.

Notation and Conventions

$\mathbb{R}^N$ is N dimensional Euclidean space, $\mathbb{R} = \mathbb{R}^1$, and $\mathbb{R}^+$ is the set $[0, +\infty[$ of positive reals. $\overline{\mathbb{R}}$ is the set $[-\infty, +\infty]$ of extended reals and $\overline{\mathbb{R}}^+$ is the set $[0, +\infty]$ of positive extended reals.

$\mathbb{Z}$ is the set of integers, $\mathbb{Z}^+$ is the set $0, 1, 2, \ldots$, and $\mathbb{Z}_n^+$ is the set $0, \ldots, n$.

The boundary of an unbounded subset of $\mathbb{R}^N$ contains the adjoined point ∞ of the one point compactification of $\mathbb{R}^N$ unless some other compactification has been specified. This boundary relative to the one point compactification of $\mathbb{R}^N$ will be called the Euclidean boundary of the set.

If ξ is a point of $\mathbb{R}^N$ and A is a subset of $\mathbb{R}^N$ the distance between ξ and A is written $|\xi - A|$.

$B(\xi, \delta)$ is the ball, in whatever metric space is specified, of center ξ and radius δ, specifically in $\mathbb{R}^N$: $B(\xi, \delta) = \{\eta : |\eta - \xi| < \delta\}$.

l_N refers to N dimensional Lebesgue measure.

If A and B are subsets of a space the set of points in A but not in B is denoted by $A - B$.

"Positive" means "≥ 0" and monotone concepts are to be taken in the wide sense, so that for example a constant function from $\mathbb{R}$ into $\mathbb{R}$ is both monotone increasing and monotone decreasing.

If D is an open subset of $\mathbb{R}^N$ the notation $\mathbb{C}^{(k)}(D)$ refers to the class of functions from D into $\mathbb{R}$ which are continuous together with their derivatives of order $\leq k$.

Limit concepts for a function f at a point do not involve the value of f at the point. Thus $\lim_{\eta \to \xi} f(\eta) = \alpha$ means that f is near α in small deleted neighborhoods of ξ.

The notation for a sequence frequently uses a dot for the index set; unless otherwise identified the index set is $\mathbb{Z}^+$, so that $A_\bullet = \{A_0, A_1, \ldots\}$.

The set on which a function f satisfies some set S of conditions is frequently denoted by $\{S\}$. Thus if f is a function from $\mathbb{R}$ into $\mathbb{R}$ the positivity set of f is $\{f \geq 0\}$.

The book is divided into three Parts. Section 1.II.3 is Section 3 of Chapter II of Part 1; in any Part, Section II.3 is Section 3 of Chapter II of that part; in any Chapter, Section 3 is Section 3 of that Chapter, and so on.