Graduate Texts in Mathematics 163

Springer-Science+Business Media, LLC

Graduate Texts in Mathematics

1 TAKEUTI/ZARING. Introduction to Axiomatic Set Theory. 2nd ed.
2 OXTOBY. Measure and Category. 2nd ed.
3 SCHAEFER. Topological Vector Spaces.
4 HILTON/STAMMBACH. A Course in Homological Algebra.
5 MAC LANE. Categories for the Working Mathematician.
6 HUGHES/PIPER. Projective Planes.
7 SERRE. A Course in Arithmetic.
8 TAKEUTI/ZARING. Axiomatic Set Theory.
9 HUMPHREYS. Introduction to Lie Algebras and Representation Theory.
10 COHEN. A Course in Simple Homotopy Theory.
11 CONWAY. Functions of One Complex Variable I. 2nd ed.
12 BEALS. Advanced Mathematical Analysis.
13 ANDERSON/FULLER. Rings and Categories of Modules. 2nd ed.
14 GOLUBITSKY/GUILLEMIN. Stable Mappings and Their Singularities.
15 BERBERIAN. Lectures in Functional Analysis and Operator Theory.
16 WINTER. The Structure of Fields.
17 ROSENBLATT. Random Processes. 2nd ed.
18 HALMOS. Measure Theory.
19 HALMOS. A Hilbert Space Problem Book. 2nd ed.
20 HUSEMOLLER. Fibre Bundles. 3rd ed.
21 HUMPHREYS. Linear Algebraic Groups.
22 BARNES/MACK. An Algebraic Introduction to Mathematical Logic.
23 GREUB. Linear Algebra. 4th ed.
24 HOLMES. Geometric Functional Analysis and Its Applications.
25 HEWITT/STROMBERG. Real and Abstract Analysis.
26 MANES. Algebraic Theories.
27 KELLEY. General Topology.
28 ZARISKI/SAMUEL. Commutative Algebra. Vol.I.
29 ZARISKI/SAMUEL. Commutative Algebra. Vol.II.
30 JACOBSON. Lectures in Abstract Algebra I. Basic Concepts.
31 JACOBSON. Lectures in Abstract Algebra II. Linear Algebra.
32 JACOBSON. Lectures in Abstract Algebra III. Theory of Fields and Galois Theory.
33 HIRSCH. Differential Topology.
34 SPITZER. Principles of Random Walk. 2nd ed.
35 WERMER. Banach Algebras and Several Complex Variables. 2nd ed.
36 KELLEY/NAMIOKA et al. Linear Topological Spaces.
37 MONK. Mathematical Logic.
38 GRAUERT/FRITZSCHE. Several Complex Variables.
39 ARVESON. An Invitation to C^*-Algebras.
40 KEMENY/SNELL/KNAPP. Denumerable Markov Chains. 2nd ed.
41 APOSTOL. Modular Functions and Dirichlet Series in Number Theory. 2nd ed.
42 SERRE. Linear Representations of Finite Groups.
43 GILLMAN/JERISON. Rings of Continuous Functions.
44 KENDIG. Elementary Algebraic Geometry.
45 LOÈVE. Probability Theory I. 4th ed.
46 LOÈVE. Probability Theory II. 4th ed.
47 MOISE. Geometric Topology in Dimensions 2 and 3.
48 SACHS/WU. General Relativity for Mathematicians.
49 GRUENBERG/WEIR. Linear Geometry. 2nd ed.
50 EDWARDS. Fermat's Last Theorem.
51 KLINGENBERG. A Course in Differential Geometry.
52 HARTSHORNE. Algebraic Geometry.
53 MANIN. A Course in Mathematical Logic.
54 GRAVER/WATKINS. Combinatorics with Emphasis on the Theory of Graphs.
55 BROWN/PEARCY. Introduction to Operator Theory I: Elements of Functional Analysis.
56 MASSEY. Algebraic Topology: An Introduction.
57 CROWELL/FOX. Introduction to Knot Theory.
58 KOBLITZ. p-adic Numbers, p-adic Analysis, and Zeta-Functions. 2nd ed.
59 LANG. Cyclotomic Fields.
60 ARNOLD. Mathematical Methods in Classical Mechanics. 2nd ed.

continued after index

John D. Dixon
Brian Mortimer

Permutation Groups

Springer

John D. Dixon
Brian Mortimer
Department of Mathematics and Statistics
Carleton University
Ottawa, Ontario
Canada K1S 5B6

With 10 Figures

Mathematics Subject Classification (1991): 20-01, 20Bxx

Library of Congress Cataloging-in-Publication Data
Dixon, John D.
Permutation groups / John D. Dixon, Brian Mortimer.
p. cm. – (Graduate texts in mathematics ; 163)
Includes bibliographical references and index.
ISBN 978-1-4612-6885-7 ISBN 978-1-4612-0731-3 (eBook)
DOI 10.1007/978-1-4612-0731-3
1. Permutation groups. I. Mortimer, Brian. II. Title.
III. Series.
QA175.D59 1996
512′.2–dc20 95-44880

Printed on acid-free paper.

Production managed by Bill Imbornoni; manufacturing supervised by Joe Quatela.
Photocomposed pages prepared from the authors' TeX files.

9 8 7 6 5 4 3 2

ISBN 978-1-4612-6885-7 SPIN 10567787

Preface

Permutation groups arguably form the oldest part of group theory. Their study dates back to the early years of the nineteenth century and, indeed, for a long time groups were always understood to be permutation groups. Although, of course, this is no longer true, permutation groups continue to play an important role in modern group theory through the ubiquity of group actions and the concrete representations which permutation groups provide for abstract groups. Today, both finite and infinite permutation groups are lively topics of research.

In this book we have tried to present something of the sweep of the development of permutation groups, explaining where the problems have come from as well as how they have been solved. Where appropriate we deal with finite and infinite groups together. Some of the theorems we consider arose in the last century or the earlier parts of this century, but most of the book deals with work done over the last few decades. In particular, the kinds of problem in finite permutation groups which can be usefully tackled has completely changed since the classification of finite simple groups was announced in 1979 (see Appendix A). One chapter of this book is devoted to the proof of the pivotal O'Nan–Scott Theorem which links the classification of finite simple groups directly to problems in finite permutation groups. We have described some of the applications of the O'Nan–Scott Theorem, even though in many cases the proofs are too technical for consideration here.

This book is intended as an introduction to permutation groups. It can be used as a text for a graduate or advanced undergraduate level course, or for independent study. The reader should have had a general introduction to group theory, and know about such things as the Sylow theorems, composition series and automorphism groups, but we have kept the prerequisites modest and recall specific facts as needed. Material in the first three chapters of the book is basic, but later chapters can be read largely independently of one another, so the text can be adapted for a variety of courses. An instructor should first cover Chapters 1 to 3 and then select material from further chapters depending on the interests of the class and the time available.

Our own experiences in learning have led us to take considerable trouble to include a large number of examples and exercises; there are over 600 of the latter. Exercises range from simple to moderately difficult, and include results (often with hints) which are referred to later. As the subject develops, we encourage the reader to accept the invitation of becoming involved in the process of discovery by working through these exercises. Keep in mind Shakespeare's advice: "Things done without example, in their issue are to be fear'd" (*King Henry the Eighth, I.ii.90*).

Although it has been a very active field during the past 20 to 30 years, no general introduction to permutation groups has appeared since H. Wielandt's influential book *Finite Permutation Groups* was published in 1964. This is a pity since the area is both interesting and accessible. Our book makes no attempt to be encyclopedic and some choices have been a little arbitrary, but we have tried to include topics indicative of the current development of the subject. Each chapter ends with a short section of notes and a selection of references to the extensive literature; again there has been no attempt to be exhaustive and many important papers have had to be omitted.

We have personally known a great deal of pleasure as our understanding of this subject has grown. We hope that some of this pleasure is reflected in the book, and will be evident to the reader. A book like this owes a clear debt to the many mathematicians who have contributed to the subject; especially Camille Jordan (whose *Traité de substitutions et des équations algébriques* was the first text book on the subject) and Helmut Wielandt, but also, more personally, to Peter Neumann and Peter Cameron. We thank Bill Kantor, Joachim Neubüser and Laci Pyber who each read parts of an early version of the manuscript and gave useful advice. Although we have taken considerable care over the manuscript, we expect that inevitably some errors will remain; if you find any, we should be grateful to hear from you.

Finally, we thank our families who have continued to support and encourage us in this project over a period of more than a decade.

Acknowledgement. The tables in Appendix B were originally published as Tables 2, 3 and 4 of: John D. Dixon and Brian Mortimer, Primitive permutation groups of degree less than 1000, *Math. Proc. Cambridge Phil. Soc.* 103 (1988) 213–238. They are reprinted with permission of Cambridge University Press.

Contents

Preface v

Notation xi

1. The Basic Ideas 1

1.1. Symmetry 1
1.2. Symmetric Groups 2
1.3. Group Actions 5
1.4. Orbits and Stabilizers 7
1.5. Blocks and Primitivity 11
1.6. Permutation Representations and Normal Subgroups 17
1.7. Orbits and Fixed Points 24
1.8. Some Examples from the Early History of Permutation Groups 28
1.9. Notes 31

2. Examples and Constructions 33

2.1. Actions on k-tuples and Subsets 33
2.2. Automorphism Groups of Algebraic Structures 35
2.3. Graphs 37
2.4. Relations 40
2.5. Semidirect Products 44
2.6. Wreath Products and Imprimitive Groups 45
2.7. Primitive Wreath Products 49
2.8. Affine and Projective Groups 52
2.9. The Transitive Groups of Degree at Most 7 58
2.10. Notes 63

3. The Action of a Permutation Group 65

3.1. Introduction 65

3.2. Orbits of the Stabilizer 66
3.3. Minimal Degree and Bases 76
3.4. Frobenius Groups 85
3.5. Permutation Groups Which Contain a Regular Subgroup 91
3.6. Computing in Permutation Groups 100
3.7. Notes 104

4. The Structure of a Primitive Group 106

4.1. Introduction 106
4.2. Centralizers and Normalizers in the Symmetric Group 107
4.3. The Socle 111
4.4. Subnormal Subgroups and Primitive Groups 115
4.5. Constructions of Primitive Groups with Nonregular Socles 119
4.6. Finite Primitive Groups with Nonregular Socles 125
4.7. Primitive Groups with Regular Socles 130
4.8. Applications of the O'Nan–Scott Theorem 137
4.9. Notes 141

5. Bounds on Orders of Permutation Groups 143

5.1. Orders of Elements 143
5.2. Subgroups of Small Index in Finite Alternating and Symmetric Groups 147
5.3. The Order of a Simply Primitive Group 151
5.4. The Minimal Degree of a 2-transitive Group 155
5.5. The Alternating Group as a Section of a Permutation Group 159
5.6. Bases and Orders of 2-transitive Groups 164
5.7. The Alternating Group as a Section of a Linear Group 168
5.8. Small Subgroups of S_n 173
5.9. Notes 175

6. The Mathieu Groups and Steiner Systems 177

6.1. The Mathieu Groups 177
6.2. Steiner Systems 178
6.3. The Extension of $AG_2(3)$ 185
6.4. The Mathieu Groups M_{11} and M_{12} 189
6.5. The Geometry of $PG_2(4)$ 192
6.6. The Extension of $PG_2(4)$ and the Group M_{22} 197
6.7. The Mathieu Groups M_{23} and M_{24} 201
6.8. The Geometry of W_{24} 205
6.9. Notes 209

7. Multiply Transitive Groups **210**

7.1. Introduction 210
7.2. Normal Subgroups 213
7.3. Limits to Multiple Transitivity 218
7.4. Jordan Groups 219
7.5. Transitive Extensions 229
7.6. Sharply k-transitive Groups 235
7.7. The Finite 2-transitive Groups 243
7.8. Notes 253

8. The Structure of the Symmetric Groups **255**

8.1. The Normal Structure of $Sym(\Omega)$ 255
8.2. The Automorphisms of $Sym(\Omega)$ 259
8.3. Subgroups of $FSym(\Omega)$ 261
8.4. Subgroups of Small Index in $Sym(\Omega)$ 265
8.5. Maximal Subgroups of the Symmetric Groups 268
8.6. Notes 273

9. Examples and Applications of Infinite Permutation Groups **274**

9.1. The Construction of a Finitely Generated Infinite p-group 274
9.2. Groups Acting on Trees 277
9.3. Highly Transitive Free Subgroups of the Symmetric Group 284
9.4. Homogeneous Groups 286
9.5. Automorphisms of Relational Structures 290
9.6. The Universal Graph 296
9.7. Notes 300

Appendix A. Classification of Finite Simple Groups **302**

Appendix B. The Primitive Permutation Groups of Degree Less than 1000 **305**

References **327**

Index **341**

Notation

$\mathbb{N}, \mathbb{Z}$	natural numbers and integers
$\mathbb{Q}, \mathbb{R}, \mathbb{C}$	rational, real and complex numbers
$\mathbb{F}_q$	field with q elements
K^d	vector space of dimension d over K
$AG_d(K), AG_d(q)$	affine geometry over K and over $\mathbb{F}_q$
$PG_d(K), PG_d(q)$	projective geometry over K and over $\mathbb{F}_q$
$S(t, k, v)$	Steiner system
$Sym(\Omega), Alt(\Omega)$	symmetric and alternating groups on Ω
S_n, A_n	symmetric and alternating groups of degree n
$FSym(\Omega)$	finitary symmetric group
C_n	cyclic group of order n
$GL_d(K), SL_d(K), \Gamma L_d(K)$	linear groups over K
$AGL_d(K), ASL_d(K), A\Gamma L_d(K)$	affine groups over K
$PGL_d(K), PSL_d(K), P\Gamma L_d(K)$	projective groups over K
$Sp_{2m}(K), Sp_{2m}(2)$	symplectic groups over K
$PGU_3(q), PSU_3(q), P\Gamma U_d(q)$	unitary groups over K
$Sz(2^s)$ and $R(3^s)$	Suzuki and Ree groups
$M_{10}, \ldots, M_{24}$	Mathieu groups
$W_{10}, \ldots, W_{24}$	Witt geometries
$\mathrm{fix}(x), \mathrm{supp}(x)$	set of fixed points and support of x
$\Omega^{\{k\}}, \Omega^{(k)}$	sets of k-subsets and k-tuples from Ω
$\mathrm{Orb}(K, \Delta)$	set of orbits of K on Δ
$\mathrm{Graph}(\Delta)$	orbital graph
$\mathrm{GCD}(m, n)$	greatest common divisor of m and n
$\lfloor x \rfloor$	largest integer $\leq x$
$\|S\|$	cardinality of set S
$\Omega \setminus \Delta$	elements of Ω not in Δ
$\Gamma \ominus \Delta$	symmetric difference of Γ and Δ
$\mathrm{Fun}(\Gamma, \Delta)$	set of functions from Γ to Δ
$\mathrm{Im}(\Phi), \ker(\Phi)$	image and kernel of Φ

$\mathrm{Aut}(X)$	automorphism group of X
$\mathrm{Inn}(G)$	inner automorphism group of G
$\mathrm{Out}(G)$	outer automorphism group of G
$\mathrm{soc}(G)$	socle of G
$N_G(H)$	normalizer of H in G
$C_G(H)$	centralizer of H in G
$H \leq G, N \lhd G$	subgroup, normal subgroup
$G \times H, G^m$	direct product, direct power
$G \rtimes H$	semidirect product
$G \, wr_\Gamma \, H$	wreath product
$G.H, G.n$	an extension of G by H, by C_n
$G : H$	a split extension of G by H