



The Costa surface

# Grundlehren der mathematischen Wissenschaften 295

*A Series of Comprehensive Studies in Mathematics*

## *Editors*

M. Artin S. S. Chern J. Coates J. M. Fröhlich  
H. Hironaka F. Hirzebruch L. Hörmander S. MacLane  
C. C. Moore J. K. Moser M. Nagata W. Schmidt  
D. S. Scott Ya. G. Sinai J. Tits M. Waldschmidt  
S. Watanabe

## *Managing Editors*

M. Berger B. Eckmann S. R. S. Varadhan

Ulrich Dierkes   Stefan Hildebrandt  
Albrecht Küster   Ortwin Wohlrab

# Minimal Surfaces I

Boundary Value Problems

With 172 Figures and 8 Colour Plates



Springer-Verlag  
Berlin Heidelberg GmbH

Ulrich Dierkes  
Stefan Hildebrandt  
Albrecht Küster  
Universität Bonn, Mathematisches Institut  
Wegelerstraße 10, D-5300 Bonn, Federal Republic of Germany

Ortwin Wohlrab  
Mauerseglerweg 3, D-5300 Bonn, Federal Republic of Germany

Mathematics Subject Classification (1991): 53 A 10, 35 J 60

Library of Congress Cataloging-in-Publication Data  
Minimal surfaces/Ulrich Dierkes... [et al.]  
v. cm. - (Grundlehren der mathematischen Wissenschaften; 295-296)  
Includes bibliographical references and indexes.  
Contents: 1. Boundary value problems - 2. Boundary regularity.

ISBN 978-3-662-02793-6 ISBN 978-3-662-02791-2 (eBook)

DOI 10.1007/978-3-662-02791-2

1. Surfaces, Minimal. 2. Boundary value problems. I. Dierkes, Ulrich. II. Series.  
QA644.M56 1992 516.3'62 - dc20 90-27155 CIP

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically those of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and a copyright fee must always be obtained from Springer-Verlag Berlin Heidelberg GmbH.  
Violations fall under the prosecution act of the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1992  
Originally published by Springer-Verlag Berlin Heidelberg New York in 1992  
Softcover reprint of the hardcover 1st edition 1992

Typesetting: Asco Trade Typesetting Ltd., Hong Kong  
41/3140-543210 Printed on acid-free paper

# Preface

This book has grown out of a series of lectures given by the second author at Bonn University. Its topic, belonging both to differential geometry and to the calculus of variations, may at first glance seem rather special. We believe, however, that it is both attractive and advantageous to consider mathematical ideas in the light of special problems, even though mathematicians nowadays tend to prefer the opposite approach, namely to emphasize general theories while relegating specific problems to play the modest role of examples. Both ways to present mathematics are equally valuable and necessary, but the theory of minimal surfaces is a good case for the first approach, to study in some detail examples which are as fascinating as they are important.

Our intention in writing this book is best characterized by a quote from Courant's treatise *Dirichlet's principle* which in several respects has been a model for our work: "*Enlightenment, however, must come from an understanding of motives; live mathematical development springs from specific natural problems which can easily be understood, but whose solutions are difficult and demand new methods of more general significance.*"

The aim of our monograph is to give an introduction to the field of two-dimensional minimal surfaces in Euclidean space with particular emphasis on boundary value problems. To keep the scope of this text within reasonable limits, we decided to omit various interesting topics altogether or to treat them only in an introductory way. In several cases we just provide surveys of the pertinent recent literature. Of the topics we regretfully had to omit we specifically mention the theory of nonparametric minimal surfaces, the investigations on stable minimal surfaces (field construction and eigenvalue criteria for the second variation of the area functional, global properties of stable surfaces) and on unstable solutions of boundary value problems as well as the recent discoveries on complete minimal surfaces. Moreover, results for surfaces of prescribed mean curvature and for minimal surfaces in Riemannian manifolds will not, with only few exceptions, be described, and geometric measure theory is not touched upon at all except for a brief survey of its applications to minimal surfaces. Nor do we provide a comprehensive presentation of the Osserman-Gulliver-Alt theory concerning the nonexistence of true and false branch points for area-minimizing solutions of boundary value problems, although we describe some of its main ideas in two instances.

Thus the reader should not expect an encyclopedic treatment of the theory of minimal surfaces but merely an introduction to the field, followed by a more thorough presentation of certain aspects which relate to boundary value problems. For further study we refer to our extensive bibliography as well as to comments and references in the Scholia attached to each chapter.

Our omissions might in part be excusable since several of the topics neglected are dealt with in the treatises of Courant [15], Federer [1], Nitsche [28], Giusti [4], and in the notes by Osserman [10], Lawson [6], Massari-Miranda [1], Struwe [11], Simon [8] and Jost [17]. The overlap of our notes and Nitsche's monograph is not very large, and in several respects the two books are complementary to each other. Together they provide a fairly comprehensive (but still not complete) picture of the theory of two-dimensional minimal surfaces.

Despite our attempts to limit the present notes in size and to restrain our desire for completeness the manuscript became too extensive for a single handy volume. Therefore we followed the advice of our publisher and split the collected material into two separate volumes (denoted as Minimal Surfaces I and II). The first one is more elementary than the second and may serve as an introduction to the theory of minimal surfaces and to Plateau's problem. To achieve a relatively self-contained presentation of our subject we have included an introductory chapter on the differential geometry of two-dimensional surfaces where we also fix the notation and collect some of the fundamental formulas to be used in later computations; it can be skipped at first reading. We have not included any material on Sobolev spaces since nowadays most students are acquainted with this topic for which our standard source of references is the treatise of Gilbarg-Trudinger [1].

The second volume is more specialized as it is mainly devoted to the study of the boundary behaviour of minimal surfaces satisfying various kinds of boundary conditions. In addition we treat the so-called thread problem, and we also provide an introduction to the general Plateau problem.

Each volume can be read independently of the other, although we use results from Volume 1 in Volume 2 and vice versa. Thus the specialist in the field may like to consult only Volume 2 whereas the nonexpert may just want to read Volume 1. However, since Part II of the first volume offers several comprehensive surveys of more recent work, this part of the book might be of interest also to the initiated.

We would like to mention that a substantial part of our notes originated in joint work of the second author with M. Grüter, E. Heinz, J.C.C. Nitsche, and F. Sauvigny. Other parts are drawn from work by H.W. Alt, R. Courant, J. Douglas, G. Dziuk, R. Gulliver, E. Heinz, W. Jäger, H. Karcher, H. Lewy, C.B. Morrey, J.C.C. Nitsche, R. Osserman, T. Radó, F. Tomi, A. Tromba, and from the authors' own results. A few items are taken from the survey of Radó and from the treatises of Courant and Nitsche.

We are grateful to many colleagues who supported us in writing this book, in particular to Maria Athanassenas, Alfred Baldes, Leung Fu Cheung, Gerhard Dziuk, David Hoffman, Jürgen Jost, Hermann Karcher, Peter Li, and Michael

Struwe. We thank Robert Osserman for providing us with Example [5] in Section 3.7. We are very much indebted to Anthony Tromba who wrote for us the main part of Chapter 11, based on his joint work [4] with Friedrich Tomi, and who also supplied material for the Scholia of Chapter 4. Martin Haneke and Andreas Wirse wrote parts of the programs that we used to draw our illustrations. We would also like to thank David Hoffman, Hermann Karcher, Konrad Polthier, and Meinhard Wohlgemuth for permitting us to use some of their drawings of complete and of periodic minimal surfaces, and Imme Haubitz for permitting us to reproduce some of her drawings of Thomsen surfaces. We are grateful to Klaus Bach, Frei Otto and Eric Pitts for providing us with photographs of various soap film experiments. The support of our work by the Computer Graphics Laboratory of the Institute of Applied Mathematics at Bonn University and of the Sonderforschungsbereich 256 was invaluable. We are especially grateful to Eva Küster who polished both style and grammar of these notes, to Carol and John Weston for advice concerning the usage of the English language, and to Anke Thiedemann who professionally and with untiring patience typed many versions of our manuscript. We should also like to thank many students and colleagues at Bonn University who pointed out errors and misprints at a first stage of the manuscript. In particular we thank Hellai Abdullah, Julia Brettschneider, John Duggan, Christoph Hamburger, Katrin Rhode and Gudrun Turowski. We acknowledge the help of David Hoffman, Hermann Karcher, Friedrich Tomi and Anthony Tromba in reading parts of the galley proofs. Without the generous support of SFB 72 and of SFB 256 at Bonn University this book could not have been written. Last but not least we should like to thank the patient publisher and his collaborators, in particular Joachim Heinze and Karl-Friedrich Koch, for their encouragement and help.

Bonn, April 30, 1991

Ulrich Dierkes, Stefan Hildebrandt  
Albrecht Küster, Ortwin Wohlrab

# Contents of Minimal Surfaces I

Introduction .....	1
--------------------	---

## **Part I. Introduction to the Geometry of Surfaces and to Minimal Surfaces**

Chapter 1. Differential Geometry of Surfaces in Three-Dimensional Euclidean Space .....	6
--	---

1.1 Surfaces in Euclidean Space .....	7
1.2 Gauss Map, Weingarten Map. First, Second, and Third Fundamental Form. Mean Curvature and Gauss Curvature .....	11
1.3 Gauss's Representation Formula, Christoffel Symbols, Gauss-Codazzi Equations, Theorema Egregium, Minding's Formula for the Geodesic Curvature .....	25
1.4 Conformal Parameters. Gauss-Bonnet Theorem .....	34
1.5 Covariant Differentiation. The Beltrami Operator .....	40
1.6 Scholia .....	48
1. Textbooks. 2. Annotations to the History of the Theory of Surfaces. 3. References to the Sources of Differential Geometry and to the Literature on Its History.	

Chapter 2. Minimal Surfaces .....	53
-----------------------------------	----

2.1 First Variation of Area. Minimal Surfaces .....	54
2.2 Nonparametric Minimal Surfaces .....	58
2.3 Conformal Representation and Analyticity of Nonparametric Minimal Surfaces .....	61
2.4 Bernstein's Theorem .....	65
2.5 Two Characterizations of Minimal Surfaces .....	71
2.6 Parametric Surfaces in Conformal Parameters. Conformal Representation of Minimal Surfaces. General Definition of Minimal Surfaces .....	74
2.7 A Formula for the Mean Curvature .....	77
2.8 Absolute and Relative Minima of Area .....	80
2.9 Scholia .....	85
1. References to the Literature on Nonparametric Minimal Surfaces. 2. Bernstein's Theorem. 3. Stable Minimal Surfaces. 4. Foliations by Minimal Surfaces.	



<b>Chapter 3. Representation Formulas and Examples of Minimal Surfaces</b> .....	<b>89</b>
3.1 The Adjoint Surface. Minimal Surfaces as Isotropic Curves in $\mathbb{C}^3$ . Associate Minimal Surfaces .....	90
3.2 Behaviour of Minimal Surfaces Near Branch Points .....	101
3.3 Representation Formulas for Minimal Surfaces .....	107
3.4 Björling's Problem. Straight Lines and Planar Lines of Curvature on Minimal Surfaces. Schwarzian Chains .....	120
3.5 Examples of Minimal Surfaces .....	135
1. Catenoid and Helicoid. 2. Scherk's Second Surface: The General Minimal Surface of Helicoidal Type. 3. The Enneper Surface. 4. Bour Surfaces. 5. Thomsen Surfaces. 6. Scherk's First Surface. 7. The Henneberg Surface. 8. Catalan's Surface. 9. Schwarz's Surface.	
3.6 Complete Minimal Surfaces .....	175
3.7 Omissions of the Gauss Map of Complete Minimal Surfaces .....	181
3.8 Scholia .....	192
1. Historical Remarks and References to the Literature. 2. Complete Minimal Surfaces of Finite Total Curvature and of Finite Topology. 3. Complete Properly Immersed Minimal Surfaces. 4. Construction of Minimal Surfaces. 5. Triply Periodic Minimal Surfaces.	

**Part II. Plateau's Problem and Free Boundary Problems**

<b>Chapter 4. The Plateau Problem and the Partially Free Boundary Problem for Minimal Surfaces</b> .....	<b>221</b>
4.1 Area Functional Versus Dirichlet Integral .....	226
4.2 Rigorous Formulation of Plateau's Problem and of the Minimization Process .....	231
4.3 Existence Proof, Part I: Solution of the Variational Problem .....	234
4.4 The Courant-Lebesgue Lemma .....	239
4.5 Existence Proof, Part II: Conformality of Minimizers of the Dirichlet Integral .....	242
4.6 Variant of the Existence Proof. The Partially Free Boundary Problem .....	253
4.7 Boundary Behaviour of Minimal Surfaces with Rectifiable Boundaries .....	259
4.8 Reflection Principles .....	267
4.9 Uniqueness and Nonuniqueness Questions .....	270
4.10 Scholia .....	276
1. Historical Remarks and References to the Literature. 2. Branch Points. 3. Embedded Solutions of Plateau's Problem. 4. More on Uniqueness and Nonuniqueness. 5. Index Theorems, Generic Finiteness, and Morse-Theory Results. 6. Obstacle Problems. 7. Systems of Minimal Surfaces.	

Chapter 5. Minimal Surfaces with Free Boundaries .....	303
5.1 Surfaces of Class $H_2^1$ and Homotopy Classes of Their Boundary Curves. Nonsolvability of the Free Boundary Problem with Fixed Homotopy Type of the Boundary Traces .....	305
5.2 Classes of Admissible Functions. Linking Condition .....	318
5.3 Existence of Minimizers for the Free Boundary Problem .....	321
5.4 Stationary Minimal Surfaces with Free or Partially Free Boundaries and the Transversality Condition .....	328
5.5 Necessary Conditions for Stationary Minimal Surfaces .....	335
5.6 Existence of Stationary Minimal Surfaces in a Simplex .....	339
5.7 Stationary Minimal Surfaces of Disk-Type in a Sphere .....	341
5.8 Report on the Existence of Stationary Minimal Surfaces in Convex Bodies .....	343
5.9 Nonuniqueness of Solutions to a Free Boundary Problem. Families of Solutions .....	345
5.10 Scholia .....	365
 Chapter 6. Enclosure Theorems and Isoperimetric Inequalities for Minimal Surfaces with Fixed or Free Boundaries .....	 367
6.1 Applications of the Maximum Principle and Nonexistence of Multiply Connected Minimal Surfaces with Prescribed Boundaries ..	368
6.2 Touching H-Surfaces and Enclosure Theorems. Further Nonexistence Results .....	372
6.3 Isoperimetric Inequalities .....	382
6.4 Estimates for the Length of the Free Trace .....	396
6.5 Scholia .....	420
1. The Isoperimetric Problem. Historical Remarks and References to the Literature. 2. Experimental Proof of the Isoperimetric Inequality. 3. Estimates for the Length of the Free Trace. 4. Enclosure Theorems and Nonexistence.	
 Bibliography .....	 427
Index of Names .....	483
Subject Index .....	486
Index of Illustrations	
Minimal Surfaces I .....	501
Minimal Surfaces II .....	506
Sources of Illustrations of Minimal Surfaces I .....	508
Colour Plates I–VIII .....	after page 218

# Contents of Minimal Surfaces II

Introduction .....	1
--------------------	---

## Part III. Boundary Behaviour of Minimal Surfaces

Chapter 7. The Boundary Regularity of Minimal Surfaces .....	6
7.1 Potential-Theoretic Preparations .....	7
7.2 Solutions of Differential Inequalities .....	21
7.3 The Boundary Regularity of Minimal Surfaces Bounded by Jordan Arcs .....	33
7.4 The Boundary Behaviour of Minimal Surfaces at Their Free Boundary: A Survey of the Results and an Outline of Their Proofs .	43
7.5 Hölder Continuity for Minima .....	48
7.6 Hölder Continuity for Stationary Surfaces .....	60
7.7 $C^{1,1/2}$ -Regularity .....	83
7.8 Higher Regularity in Case of Support Surfaces with Empty Boundaries. Analytic Continuation Across a Free Boundary .....	102
7.9 A Different Approach to Boundary Regularity .....	109
7.10 Asymptotic Expansion of Minimal Surfaces at Boundary Branch Points and Geometric Consequences .....	117
7.11 The Gauss-Bonnet Formula for Branched Minimal Surfaces .....	121
7.12 Scholia .....	128
Chapter 8. Singular Boundary Points of Minimal Surfaces .....	141
8.1 The Method of Hartman and Wintner, and Asymptotic Expansions at Boundary Branch Points .....	142
8.2 A Gradient Estimate at Singularities Corresponding to Corners of the Boundary .....	163
8.3 Minimal Surfaces with Piecewise Smooth Boundary Curves and Their Asymptotic Behaviour at Corners .....	173
8.4 An Asymptotic Expansion for Solutions of the Partially Free Boundary Problem .....	186
8.5 Scholia .....	196

Chapter 9. Minimal Surfaces with Supporting Half-Planes ..... 198

9.1 An Experiment ..... 199

9.2 Examples of Minimal Surfaces with Cusps  
on the Supporting Surface ..... 202

9.3 Set-up of the Problem. Properties of Stationary Solutions ..... 206

9.4 Classification of the Contact Sets ..... 208

9.5 Nonparametric Representation, Uniqueness, and Symmetry  
of Solutions ..... 213

9.6 Asymptotic Expansions for Surfaces of Cusp-Types I and III.  
Minima of Dirichlet's Integral ..... 216

9.7 Asymptotic Expansions for Surfaces of the Tongue/Loop-Type II . 218

9.8 Final Results on the Shape of the Trace. Absence of Cusps.  
Optimal Boundary Regularity ..... 221

9.9 Proof of the Representation Theorem ..... 223

9.10 Scholia ..... 229

1. Remarks about Chapter 9. 2. Numerical Solutions. 3. Another Uniqueness  
Theorem for Minimal Surfaces with a Semifree Boundary.

**Part IV. Ramifications: The Thread Problem. The General Plateau Problem**

Chapter 10. The Thread Problem ..... 250

10.1 Experiments and Examples. Mathematical Formulation  
of the Simplest Thread Problem ..... 250

10.2 Existence of Solutions to the Thread Problem ..... 255

10.3 Analyticity of the Movable Boundary ..... 271

10.4 Scholia ..... 291

Chapter 11. The General Problem of Plateau ..... 293

11.1 The General Problem of Plateau. Formulation and Examples .... 293

11.2 A Geometric Approach to Teichmüller Theory of Oriented Surfaces 299

11.3 Symmetric Riemann Surfaces and Their Teichmüller Spaces ..... 307

11.4 The Mumford Compactness Theorem ..... 315

11.5 The Variational Problem ..... 319

11.6 Existence Results for the General Problem of Plateau in  $\mathbb{R}^3$  ..... 328

11.7 Scholia ..... 339

Bibliography ..... 341

Index of Names ..... 397

Subject Index ..... 400

Index of Illustrations

    Minimal Surfaces II ..... 415

    Minimal Surfaces I ..... 417

Sources of Illustrations of Minimal Surfaces II ..... 422