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Editors

Nonstandard Analysis in Practice

With 34 Figures



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The cover picture shows the graph of a solution's real part of the Liouville differential equation (see Fig. 2.3) produced by A. Fruchard.

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