

Contents

Preface	xiii
-------------------	------

CHAPTER 0 PRELIMINARIES

1. Green's Theorem	1
1.1. Differential Operators and Adjoints	1
2. The Continuity Equation	2
2.1. The Heat Equation and the Laplace Equation	5
3. A Model for the Vibrating String	6
4. Small Vibrations of a Membrane	8
5. Transmission of Sound Waves	11
6. The Navier–Stokes System	13
7. The Euler Equations	14
8. Isentropic Potential Flows	14
8.1. Steady Potential Isentropic Flows	16
9. Partial Differential Equations	16
Problems and Complements	
1. The Theorem of Ascoli–Arzelà	17
2. The Cauchy–Schwarz–Young Inequality	19
3. The Hölder Inequality	20
4. The Minkowski Inequality	22
5. Continuity in $L^p(\Omega)$	24
6. Mollifiers	26

CHAPTER I QUASI-LINEAR EQUATIONS AND THE CAUCHY–KOWALEWSKI THEOREM

1. Quasi-Linear Second-Order Equations in Two Variables	29
2. Characteristics and Singularities	31
2.1. Coefficients Independent of u_x and u_y	32
3. Quasi-Linear Second-Order Equations	33
3.1. Constant Coefficients	35
3.2. Variable Coefficients	36
4. Quasi-Linear Equations of Order $m \geq 1$	36
4.1. Characteristic Surfaces	38

5. Analytic Data and the Cauchy–Kowalewski Theorem	38
5.1. Reduction to the Normal Form	39
6. Proof of the Cauchy–Kowalewski Theorem	41
6.1. Estimating the Derivatives of \mathbf{u} at the Origin	42
7. Auxiliary Inequalities	43
8. Auxiliary Estimations at the Origin	45
9. Proof of the Cauchy–Kowalewski Theorem Concluded	47
9.1. Proof of Lemma 6.1	47
 Problems and Complements	
1. Quasi-Linear Second-Order Equations in Two Variables	48
2. Characteristics and Singularities	48
5. The Cauchy–Kowalewski Theorem	49
6. Proof of the Cauchy–Kowalewski Theorem	49
8. The Generalized Leibniz Rule	49
9. Proof of the Cauchy–Kowalewski Theorem Concluded	50

CHAPTER II

THE LAPLACE EQUATION

1. Preliminaries	45
1.1. Boundary Value Problems	51
1.2. The Cauchy Problem	52
1.3. Well-Posedness and a Counterexample of Hadamard	53
1.4. Radial Solution	54
2. Green and Stokes Identities	55
2.1. Stokes Identities	56
3. Green’s Function and the Dirichlet Problem for the Sphere	57
3.1. Green’s Function for a Sphere	60
4. Subharmonic Functions and the Mean Value Property	62
4.1. The Maximum Principle	65
4.2. Structure of Subharmonic Functions	66
5. Estimating Harmonic Functions and Their Derivatives	68
5.1. The Harnack Inequality and the Liouville Theorem	68
5.2. Analyticity of Harmonic Functions	70
6. The Dirichlet Problem	72
7. About the Exterior Sphere Condition	76
7.1. The Case $N = 2$ and $\partial\Omega$ Piecewise Smooth	77
7.2. A Counterexample of Lebesgue for $N = 3$	77
8. The Poisson Integral for the Half Space	79
9. Schauder Estimates of Newtonian Potentials	81
10. Potential Estimates in $L^p(\Omega)$	85
11. Local Solutions	88
11.1. Local Weak Solutions	89

12. Nonhomogeneous Problems	90
12.1. The Case $f \in C_0^\infty(\Omega)$	92
12.2. The Case $f \in C^n(\Omega)$	92
Problems and Complements	93
1. Preliminaries	93
2. Green and Stokes Identities	95
3. Green's Function and the Dirichlet Problem for the Sphere	95
4. Subharmonic Functions and the Mean Value Property	96
5. Estimating Harmonic Functions	103
7. Barriers	107
8. Problems in Unbounded Domains	108
9. Schauder Estimates	109
10. Potential Estimates in $L^p(\Omega)$ and Hardy's Inequality	110
11. Local Solutions	115
12. Nonhomogeneous Problems	115

CHAPTER III

THE DOUBLE LAYER POTENTIAL AND BOUNDARY VALUE PROBLEMS

1. The Double Layer Potential	116
2. On the Integral Defining the Double Layer Potential	118
3. The Jump Condition of $W(\partial\Omega, x_o; v)$ across $\partial\Omega$	120
4. More on the Jump Condition across $\partial\Omega$	123
5. The Normal Derivative of $W(\partial\Omega, x_o; v)$ across $\partial\Omega$	126
6. The Dirichlet Problem by Integral Equations	131
7. The Neumann Problem by Integral Equations	132
8. Green's Function for the Neumann Problem	134
8.1. Finding $\mathcal{G}(\cdot; \cdot)$	136
9. Eigenvalue Problems	138
9.1. About the Kernel $\mathcal{G}(\cdot; \cdot)$	138
10. Compact Kernels in $L^2(\Omega)$	139
10.1. Proof of Lemma 10.1	140
10.2. Proof of Lemma 10.2	141
11. Compact Kernels in $L^2(\Omega)$ and Green's Function	144
12. More on Compactness and the Eigenvalue Problem	146
Problems and Complements	
2. On the Integral Defining the Double Layer Potential	150
6. The Dirichlet Problem by Integral Equations	150
7. The Neumann Problem by Integral Equations	151
8. The Green Function for the Neumann Problem	152
9. Eigenvalue Problems	154
10. Metric Spaces and Compactness	154

CHAPTER IV

INTEGRAL EQUATIONS
AND EIGENVALUE PROBLEMS

1. Integral Equations	161
2. The Homogeneous and Adjoint Equations	162
2.1. The Case of $(1.1)'$	163
3. Existence of Solutions for Small λ	163
3.1. Existence of Solutions for Small λ , for $(1.1)'$	165
4. Separable Kernels	166
4.1. Solving the Homogeneous Equations	167
4.2. Solving the Inhomogeneous Equation	168
4.3. Separable Kernels and the Integral Equation $(1.1)'$	169
5. Small Perturbations of Separable Kernels	171
5.1. Perturbations of Separable Kernels in $\partial\Omega$	174
6. General Kernels	176
6.1. Potential Kernels	176
6.2. General Kernels Supported on $\partial\Omega$	179
7. Applications to the Neumann Problem	181
7.1. An Auxiliary Lemma	182
7.2. Proof of Lemma 7.1	183
8. Integral Equations and Operators in $L^2(\Omega)$	183
8.1. Kernels Acting on $L^2(\Omega)$	184
8.2. Integral Equations in $L^2(\Omega)$	185
8.3. The Related Operators and Their Norms	186
8.4. Compact Symmetric Kernels and Operators	187
9. Solving Integral Equations in $L^2(\Omega)$	187
9.1. Existence of Solutions for Small λ	188
9.2. Separable Kernels	191
9.3. Almost Separable Kernels	193
10. The Eigenvalue Problem	194
11. The First Eigenvalue and Eigenfunction	197
11.1. Proof of Theorem 11.1 Concluded	199
12. The Sequence of Eigenvalues	200
13. More on the Sequence of the Eigenvalues	201
14. Questions of Completeness and the Hilbert–Schmidt Theorem	203
14.1. The Case of $K(x; \cdot) \in L^2(\Omega)$ Uniformly in x	205
15. The Eigenvalue Problem for the Laplacian	207
15.1. An Expansion of the Green Function	209
Problems and Complements	
1. Integral Equations	209
2. The Homogeneous and Adjoint Equations	213
3. Existence of Solutions for Small λ	213
4. Separable Kernels	214
6. General Kernels and the Weierstrass Theorem	215

7. Applications to the Neumann Problem	219
10. The Eigenvalue Problem	220
11. The First Eigenvalue	220
12. The Sequence of Eigenvalues	220
14. Questions of Completeness	221
15. The Eigenvalue Problem for the Laplacian	223

CHAPTER V
THE HEAT EQUATION

1. Preliminaries	225
2. The Cauchy Problem by Similarity Solutions	226
2.1. The Backward Cauchy Problem	230
3. The Maximum Principle and Uniqueness (Bounded Domains)	231
3.1. A Priori Estimates	232
3.2. Ill-Posed Problems	232
3.3. Uniqueness (Bounded Domains)	233
4. The Maximum Principle in \mathbf{R}^N	233
4.1. A Priori Estimates	236
4.2. About the Growth Conditions (4.4) and (4.7)	237
5. Uniqueness of Solutions to the Cauchy Problem	237
5.1. A Counterexample	238
6. Initial Data in $L^1_{\text{loc}}(\mathbf{R}^N)$	240
6.1. Initial Data in the Sense of $L^1_{\text{loc}}(\mathbf{R}^N)$	244
7. Remarks on the Cauchy Problem	245
7.1. About Regularity	245
7.2. Instability of the Backward Problem	246
8. Estimates near $t = 0$	246
9. The Nonhomogeneous Cauchy Problem	249
10. Problems in Bounded Domains	251
10.1. The Strong Solution	254
10.2. The Weak Solution and Energy Inequalities	254
11. Energy and Logarithmic Convexity	256
11.1. Uniqueness for Some Ill-Posed Problems	258
12. Local Solutions	258
12.1. Variable Cylinders	263
12.2. The Case $ \alpha = 0$	264
13. The Harnack Inequality	264
13.1. Compactly Supported Subsolutions	266
13.2. Proof of Theorem 13.1	267
14. Positive Solutions in S_T	269
14.1. Nonnegative Solutions	271
Problems and Complements	
2. Similarity Methods	274

3. The Maximum Principle in Bounded Domains	280
4. The Maximum Principle in \mathbf{R}^N	284
5. Uniqueness of Solutions to the Cauchy Problem	286
6. Initial Data in $L^1_{\text{loc}}(\mathbf{R}^N)$	286
7. Remarks on the Cauchy Problem	286
10. Problems in Bounded Domains	287
11. Energy and Logarithmic Convexity	288
12. On the Local Behavior of Solutions	288
13. The Harnack Inequality	291

CHAPTER VI
THE WAVE EQUATION

1. The One-Dimensional Wave Equation	292
1.1. A Property of Solutions	293
2. The Cauchy Problem	294
3. Nonhomogeneous Problems	296
4. A Boundary Value Problem (Vibrating String)	297
4.1. Separation of Variables	298
4.2. Odd Reflection	300
4.3. Energy and Uniqueness	301
4.4. Nonhomogeneous Problems	301
5. The Initial Value Problem in N Dimensions	302
5.1. Spherical Means	302
5.2. The Darboux Formula	303
5.3. An Equivalent Formulation of the Cauchy Problem	304
6. The Cauchy Problem in \mathbf{R}^3	304
7. The Cauchy Problem in \mathbf{R}^2	308
8. The Nonhomogeneous Cauchy Problem	310
9. Cauchy Problems for Nonhomogeneous Surfaces	311
9.1. Reduction to Homogeneous Data on $t = \Phi(x)$	312
9.2. The Problem with Homogeneous Data	313
10. Solutions in Half Space. The Reflexion Technique	314
10.1. An Auxiliary Problem	314
10.2. Homogeneous Data on the Hyperplane $x_3 = 0$	315
11. A Boundary Value Problem	316
12. Hyperbolic Equations in Two Variables	318
13. The Characteristic Goursat Problem	318
13.1. Proof of Theorem 13.1 (Existence)	319
13.2. Proof of Theorem 13.1 (Uniqueness)	321
14. The Non-Characteristic Problem and the Riemann Function	322
15. Symmetry of the Riemann Function	325
Problems and Complements	
1. The One-Dimensional Wave Equation	326

2. The d'Alembert Formula	327
3. Nonhomogeneous Problems	327
4. Solutions for the Vibrating String	328
6. Cauchy Problems in \mathbf{R}^3	330
7. Cauchy Problems in \mathbf{R}^2 and the Method of Descent	335
8. Nonhomogeneous Cauchy Problems	336
10. The Reflection Technique	339
11. Problems in Bounded Domains	340
12. Hyperbolic Equations in Two Variables	341
14. Goursat Problems	341

CHAPTER VII
EQUATIONS OF FIRST ORDER
AND CONSERVATION LAWS

1. Quasi-Linear Equations	343
2. The Cauchy Problem	344
2.1. The Case of Two Independent Variables	345
2.2. The Case of N Independent Variables	345
3. Solving the Cauchy Problem	346
3.1. Constant Coefficients	347
2.1. Solutions in Implicit Form	348
4. Equations in Divergence Form and Weak Solutions	349
4.1. Surfaces of Discontinuity	350
4.2. The Shock Line	351
5. The Initial Value Problem	352
5.1. Conservation Laws	352
6. Conservation Laws in One Space Dimension	353
6.1. Weak Solutions and Shocks	355
6.2. Lack of Uniqueness	356
7. Weak Solutions to (6.4) when $a(\cdot)$ Is Strictly Increasing	356
7.1. Lax Notion of Weak Solution	357
8. Constructing Weak Solution – I	359
9. Constructing Weak Solution – II	362
9.1. Estimation of I_n	364
10. The Theorems of Existence and Stability	366
10.1. Existence of Solutions	366
10.2. Stability	267
11. Proof of Theorem 10.1: The Representation Formula (10.5)	368
12. Proof of Theorem 10.1: Initial Data in the Sense of $L_{loc}^1(\mathbf{R})$	370
13. Proof of Theorem 10.1: The Weak Form of the P.D.E.	371
14. An Integral Form of (10.2)	372
15. Sup-Estimates and Invariants	375
16. More on Sup-Estimates and Invariants	378
16.1 Compactly Supported Initial Data	379

17. The Entropy Condition	380
17.1. Entropy Solutions	380
17.2. The Solution of Theorem 10.1 Is an Entropy Solution	381
17.3. Remarks on the Shock and Entropy Conditions	384
18. The Kruzhkov Uniqueness Theorem	385
18.1. Proof of the Uniqueness Theorem (I)	386
18.2. Proof of the Uniqueness Theorem (II)	388
19. The Maximum Principle for Entropy Solutions	389
20. Stability in $L^1(\mathbf{R}^N)$	390
21. Asymptotic Behavior of Solutions of Burgers' Equation	391
21.1. Constructing the Asymptotic Limit for $x \geq x^o$	393
21.2. Proof of Theorem 21.1 for $x \geq x^o$	395
21.3. Proof of Theorem 21.1	395
22. The Asymptotic Profile when $F \in C^2(\mathbf{R})$	396
22.1. Constructing the Asymptotic Limit for $x \geq x^o + a(0)t$	397
Problems and Complements	
3. Solving the Cauchy Problem	399
6. Explicit Solutions to the Burgers Equation	401
7. The Method of Viscosity for the Burgers Equation	403
8. Proof of (8.8)	405
9. Constructing Solutions	406
17. Entropy Solutions	407
18. The Uniqueness Theorem	409
Bibliography	411
Index	413