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Degenerate Parabolic Equations

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Preface

1. Elliptic equations: Harnack estimates and Hölder continuity

Considerable progress was made in the early 1950s and mid-1960s in the theory of elliptic equations, due to the discoveries of DeGiorgi [33] and Moser [81,82]. Consider local weak solutions of

$$\text{(1.1)} \qquad \begin{cases} u \in W^{1,2}_{loc}(\Omega), & \Omega \text{ a domain in } \mathbf{R}^N \\ \left(a_{ij}u_{x_i}\right)_{x_j} = 0 & \text{in } \Omega, \end{cases}$$

where the coefficients $x \rightarrow a_{ij}(x)$, $i, j = 1, 2, \ldots, N$ are assumed to be only bounded and measurable and satisfying the ellipticity condition

$$\text{(1.2)} \qquad a_{ji}\xi_i\xi_j \geq \lambda|\xi|^2, \quad \text{a.e. } \Omega, \quad \forall \xi \in \mathbf{R}^N, \quad \text{for some } \lambda > 0.$$

DeGiorgi established that local solutions are Hölder continuous and Moser proved that non-negative solutions satisfy the Harnack inequality. Such inequality can be used, in turn, to prove the Hölder continuity of solutions. Both authors worked with *linear* p.d.e.'s. However the linearity has no bearing in the proofs. This permits an extension of these results to quasilinear equations of the type

$$\text{(1.3)} \qquad \begin{cases} u \in W^{1,p}_{loc}(\Omega), & p > 1 \\ \operatorname{div} \mathbf{a}(x, u, Du) + b(x, u, Du) = 0, & \text{in } \Omega, \end{cases}$$

with structure conditions

$$\text{(1.4)} \qquad \begin{cases} \mathbf{a}(x, u, Du) \cdot Du \geq \lambda|Du|^p - \varphi(x), & \text{a.e. } \Omega_T,\ p > 1 \\ |\mathbf{a}(x, u, Du)| \leq \Lambda|Du|^{p-1} + \varphi(x), \\ |b(x, u, Du)| \leq \Lambda|Du|^{p-1} + \varphi(x). \end{cases}$$

Here $0 < \lambda \leq \Lambda$ are two given constants and $\varphi \in L^{\infty}_{loc}(\Omega)$ is non-negative. As a prototype we may take

$$\text{div}\,|Du|^{p-2}Du = 0, \qquad \text{in} \quad \Omega, \qquad p > 1. \tag{1.5}$$

The modulus of ellipticity of (1.5) is $|Du|^{p-2}$. Therefore at points where $|Du| = 0$, the p.d.e. is degenerate if $p > 2$ and it is singular if $1 < p < 2$.

By using the methods of DeGiorgi, Ladyzhenskaja and Ural'tzeva [66] established that weak solutions of (1.4) are Hölder continuous, whereas Serrin [92] and Trudinger [96], following the methods of Moser, proved that non-negative solutions satisfy a Harnack principle. The generalisation is twofold, i.e., the principal part $\mathbf{a}(x, u, Du)$ is permitted to have a *non-linear dependence* with respect to u_{x_i}, $i = 1, 2, \ldots, N$, and a *non-linear growth* with respect to $|Du|$. The latter is of particular interest since the equation in (1.5) might be either degenerate or singular.

2. Parabolic equations: Harnack estimates and Hölder continuity

The first parabolic version of the Harnack inequality is due to Hadamard [50] and Pini [86] and applies to non-negative solutions of the heat equation. It takes the following form. Let u be a non-negative solution of the heat equation in the cylindrical domain $\Omega_T \equiv \Omega \times (0, T)$, $0 < T < \infty$, and for $(x_o, t_o) \in \Omega_T$ consider the cylinder

$$Q_\rho \equiv B_\rho(x_o) \times (t_o - \rho^2, t_o], \quad B_\rho(x_o) \equiv \{|x - x_o| < \rho\}. \tag{2.1}$$

There exists a constant γ depending only upon N, such that if $Q_{2\rho} \subset \Omega_T$, then

$$u(x_o, t_o) \geq \gamma \sup_{B_\rho(x_o)} u(x, t_o - \rho^2). \tag{2.2}$$

The proof is based on local representations by means of heat potentials. A striking result of Moser [83] is that (2.2) continues to hold for non-negative weak solutions of

$$\begin{cases} u \in V^{1,2}(\Omega_T) \equiv L^\infty\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; W^{1,2}(\Omega)\right), \\ u_t - \left(a_{ij}(x,t) u_{x_i}\right)_{x_j} = 0, \qquad \text{in } \Omega_T \end{cases} \tag{2.3}$$

where $a_{ij} \in L^\infty(\Omega_T)$ satisfy the analog of the ellipticity condition (1.2). As before, it can be used to prove that weak solutions are locally Hölder continuous in Ω_T. Since the linearity of (2.3) is immaterial to the proof, one might expect, as in the elliptic case, an extension of these results to quasilinear equations of the type

$$\begin{cases} u \in V^{1,p}(\Omega_T) \equiv L^\infty\left(0, T; L^2(\Omega)\right) \cap L^p\left(0, T; W^{1,p}(\Omega)\right), \\ u_t - \text{div}\,\mathbf{a}(x, t, u, Du) = b(x, t, u, Du), \qquad \text{in } \Omega_T, \end{cases} \tag{2.4}$$

where the structure condition is as in (1.4). Surprisingly however, Moser's proof could be extended only for the case $p = 2$, i.e., for equations whose principal part has a *linear growth* with respect to $|Du|$. This appears in the work of Aronson and Serrin [7] and Trudinger [97]. The methods of DeGiorgi also could not be extended. Ladyzenskaja et al. [67] proved that solutions of (2.4) are Hölder continuous, provided the principal part has exactly a *linear growth* with respect to $|Du|$. Analogous results were established by Kruzkov [60,61,62] and by Nash [84] by entirely different methods.

Thus it appears that unlike the elliptic case, the degeneracy or singularity of the principal part plays a peculiar role, and for example, for the non-linear equation

$$u_t - \operatorname{div} |Du|^{p-2} Du = 0, \qquad \text{in } \Omega_T, \quad p > 1, \tag{2.5}$$

one could not establish whether non-negative weak solutions satisfy the Harnack estimate or whether a solution is locally Hölder continuous.

3. Parabolic equations and systems

These issues have remained open since the mid-1960s. They were revived however with the contributions of N.N. Ural'tzeva [100] in 1968 and K. Uhlenbeck [99] in 1977. Consider the system

$$\begin{cases} \mathbf{u} \equiv (u_1, u_2, \ldots, u_n), & u_i \in W^{1,p}_{loc}(\Omega),\ p > 1,\ i = 1, 2, \ldots n, \\ \operatorname{div} |D\mathbf{u}|^{p-2} Du_i = 0, & \text{in } \Omega\,. \end{cases} \tag{3.1}$$

When $p > 2$, Ural'tzeva and Uhlenbeck prove that local solutions of (3.1) are of class $C^{1,\alpha}_{loc}(\Omega)$, for some $\alpha \in (0,1)$. The parabolic version of (3.1) is

$$\begin{cases} \mathbf{u} \equiv (u_1, u_2, \ldots, u_n), & u_i \in V^{1,p}(\Omega_T),\ i = 1, 2, \ldots n, \\ u_t - \operatorname{div} |D\mathbf{u}|^{p-2} Du_i = 0, & \text{in } \Omega_T\,. \end{cases} \tag{3.2}$$

Besides their intrinsic mathematical interest, this kind of system arises from geometry [99], quasiregular mappings [2,17,55,89] and fluid dynamics [5,8,56,57,74,75]. In particular Ladyzenskaja [65] suggests systems of the type of (3.2) as a model of motion of non-newtonian fluids. In such a case $\mathbf{u}$ is the velocity vector. Non-newtonian here means that the stress tensor at each point of the fluid is *not* linearly proportional to the matrix of the space-gradient of the velocity.

The function $w = |D\mathbf{u}|^2$ is formally a subsolution of

$$\frac{\partial}{\partial t} w - \left(a_{\ell,k} w^{\frac{p-2}{2}} w_{x_k} \right)_{x_\ell} \le 0 \quad \text{in } \Omega_T, \tag{3.3}$$

where

$$a_{\ell,k} \equiv \left\{ \delta_{\ell,k} + (p-2) \frac{u_{i,x_\ell} u_{i,x_k}}{|D\mathbf{u}|^2} \right\}.$$

This is a *parabolic* version of a similar finding observed in [99,100] for elliptic systems. Therefore a *parabolic* version of the Ural'tzeva and Uhlenbeck result requires some understanding of the *local* behaviour of solutions of the porous media equation

$$u_t - \Delta u^m = 0, \qquad u \ge 0, \quad m > 0, \tag{3.4}$$

and its quasilinear versions. Such an equation is degenerate at those points of Ω_T where $u=0$ if $m>1$ and singular if $0<m<1$.

The porous medium equation has a life of its own. We only mention that questions of regularity were first studied by Caffarelli and Friedman. It was shown in [21] that *non-negative* solutions of the *Cauchy problem* associated with (3.4) are Hölder continuous. The result is not *local.*

A more *local* point of view was adopted in [20,35,90]. However these contributions could only establish that the solution is continuous with a *logarithmic* modulus of continuity.

In the mid-1980s, some progress was made in the theory of degenerate p.d.e.'s of the type of (2.5), for $p>2$. It was shown that the solutions are locally Hölder continuous (see [39]). Surprisingly, the same techniques can be suitably modified to establish the *local* Hölder continuity of *any local* solution of quasilinear porous medium-type equations. These modified methods, in turn, are crucial in proving that weak solutions of the systems (3.2) are of class $C^{1,\alpha}_{loc}(\Omega_T)$.

Therefore understanding the local structure of the solutions of (2.5) has implications to the theory of systems and the theory of equations with degeneracies quite different than (2.5).

4. Main results

In these notes we will discuss these issues and present results obtained during the past five years or so. These results follow, one way or another, from a single unifying idea which we call *intrinsic rescaling.* The diffusion process in (2.5) evolves in a time scale determined instant by instant by the solution itself, so that, loosely speaking, it can be regarded as the heat equation in its own intrinsic time-configuration. A precise description of this fact as well as its effectiveness is linked to its technical implementations.

We collect in Chap. I notation and standard material to be used as we proceed. Degenerate or singular p.d.e. of the type of (2.4) are introduced in Chap. II. We make precise their functional setting and the meaning of solutions and we derive *truncated* energy estimates for them. In Chaps. III and VI, we state and prove theorems regarding the local and global Hölder continuity of weak solutions of (2.4) both for $p>2$ and $1<p<2$ and discuss some open problems. In the singular case $1<p<2$, we introduce in Chap. IV a novel iteration technique quite different than that of DeGiorgi [33] or Moser [83].

These theorems assume the solutions to be locally or globally bounded. A theory of boundedness of solutions is developed in Chap. V and it includes equations with lower order terms exhibiting the Hadamard *natural* growth condition. The sup-estimates we prove appear to be dramatically different than those in the linear theory. Solutions are locally bounded only if they belong to $L^r_{loc}(\Omega_T)$ for some $r\geq 1$ satisfying

$$\lambda_r \equiv N(p-2)+rp>0 \tag{4.1}$$

and such a condition is sharp. In Chap. XII we give a counterexample that shows that if (4.1) is violated, then (2.5) has unbounded solutions.

The Hölder estimates and the L^∞-bounds are the basis for an organic theory of local and global behaviour of solutions of such degenerate and/or singular equations.

In Chaps. VI and VII we present an *intrinsic* version of the Harnack estimate and attempt to trace their connection with Hölder continuity. The natural parabolic cylinders associated with (2.5) are

$$Q_\rho \equiv B_\rho(x_o)\times(t_o-\rho^p,, t_o], \qquad (x_o,t_o)\in\Omega_T\,. \tag{4.2}$$

We show by counterexamples that the Harnack estimate (2.2) cannot hold for non-negative solutions of (2.5), in the geometry of (4.2). It does hold however in a time-scale intrinsic to the solution itself. These Harnack inequalities reduce to (2.2) when $p=2$. In the degenerate case $p>2$ we establish a *global* Harnack type estimate for non-negative solutions of (1.5) in the whole strip $\Sigma_T\equiv\mathbf{R}^N\times(0,T)$. We show that such an estimate is equivalent to a growth condition on the solution as $|x|\to\infty$. If $\max\{1;\frac{2N}{N+1}\}<p<2$, a surprising result is that the Harnack estimate holds in an *elliptic* form, i.e., holds over a ball B_ρ at a given time level. This is in contrast to the behaviour of non-negative solutions of the heat equation as pointed out by Moser [83] by a counterexample. These Harnack estimates in either the degenerate or singular case have been established *only* for non-negative solutions of the homogeneous equation (2.5). The proofs rely on some sort of non-linear versions of 'fundamental solutions'. It is natural to ask whether they hold for quasilinear equations. This is a challenging open problem and parallels the Hadamard [50] and Pini [86] approach via *fundamental* solutions, versus the *'non-linear'* approach of Moser [83].

The number p is required to be larger than $2N/(N+1)$ and such a condition is sharp for a Harnack estimate to hold. The case $1<p\le 2N/(N+1)$ is not fully understood and it seems to suggest questions similar to those of the limiting Sobolev exponent for elliptic equations (see Brézis [19]) and questions in differential geometry. Here we only mention that as $p\searrow 1$, (2.5) tends *formally* to a p.d.e. of the type of motion by mean curvature.

Hölder and Harnack estimates as well as precise sup-bounds coalesce in the theory of the Cauchy problem associated with (2.4). This is presented in Chap. XI for the degenerate case $p>2$ and in Chap. XII for the singular case $1<p<2$. When $p>2$, we identify the optimal growth of the initial datum as $|x|\to\infty$ for a solution, local or global in time, to exist. This is the analog of the theory of Tychonov [98], Tacklind [94] and Widder [105] for the heat equation. When $1<p<2$ it turns out that any non-negative initial datum $u_o\in L^1_{loc}(\mathbf{R}^N)$ yields a *unique* solution global in time. In general

$$|Du|\notin L^p_{loc}\left(\mathbf{R}^N\times\mathbf{R}^+\right), \qquad 1<p\le\frac{2N}{N+1}\,.$$

Therefore the main difficulty of the theory is to make precise the meaning of solution. We introduce in Chap. XII a new notion of non-negative weak solutions and establish the existence and *uniqueness* of such solutions. We show by a counterexample that these might be discontinuous. Thus, in view of the possible singularities, the notion of solution is dramatically different than the notion of *'viscosity'* solution. Issues of solutions of variable sign as well as their local and global behaviour are open.

In Chaps. VIII-X, we turn to systems of the type (3.2) and prove that

$$u^{(i)}_{x_j}\in C^\alpha_{loc}(\Omega_T), \quad i=1,2,\dots,n, \quad j=1,2,\dots,N, \tag{4.3}$$

provided $p>2N/(N+1)$. Analogous estimates are derived for all $p>1$ for solutions in $L^r_{loc}(\Omega_T)$, where $r\ge1$ satisfies (4.1). Again such a condition is sharp for (4.3) to hold. Near the lateral boundary of Ω_T we establish C^α estimates *for all* $\alpha \in (0,1)$, provided $p > \max\left\{1; \frac{2N}{N+2}\right\}$. Estimates in the class $C^{1,\alpha}$ near the boundary are still lacking even in the elliptic case.

A similar spectrum of results could be developed for equations of the type (3.4). We have avoided doing this to keep the theory as organic and unified as possible.

We have chosen not to present existence theorems for boundary value problems associated with (2.4) or (3.2). Theorems of this kind are mostly based on Galerkin approximations and appear in the literature in a variety of forms. We refer, for example, to [67] or [73]. Given the a priori estimates presented here these can be obtained alternatively by a limiting process in a family of approximating problems and an application of Minty's Lemma. These notes can be ideally divided in four parts:

1. Hölder continuity and boundedness of solutions (Chapters I-V)
2. Harnack type estimates (Chapters VI-VII)
3. Systems (Chapters VIII-X)
4. Non-negative solutions in a strip Σ_T (Chapters XI-XII).

These parts are technically linked but they are conceptually independent, in the sense that they deal with issues that have developed in independent directions. We have attempted to present them in such a way that they can be approached independently.

The motivation in writing these notes, beyond the specific degenerate and singular p.d.e., is to present a body of ideas and techniques that are surprisingly flexible and adaptable to a variety of parabolic equations bearing, in one way or another, a degeneracy or singularity.

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