A.A. Dezin
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## An Introduction to a General Theory of Linear Boundary Value Problems

Translated from the Russian
by Ralph P. Boas

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# Preface

Let me begin by explaining the meaning of the title of this book. In essence, the book studies boundary value problems for linear partial differential equations in a finite domain in $n$-dimensional Euclidean space. The problem that is investigated is the question of the dependence of the nature of the solvability of a given equation on the way in which the boundary conditions are chosen, i.e. on the supplementary requirements which the solution is to satisfy on specified parts of the boundary.

The branch of mathematical analysis dealing with the study of boundary value problems for partial differential equations is often called mathematical physics.

Classical courses in this subject usually consider quite restricted classes of equations, for which the problems have an immediate physical context, or generalizations of such problems.

With the expanding domain of application of mathematical methods at the present time, there often arise problems connected with the study of partial differential equations that do not belong to any of the classical types. The elucidation of the correct formulation of these problems and the study of the specific properties of the solutions of similar equations are closely related to the study of questions of a general nature.

Among these are the following:

1. What accounts for the special position of the classical equations of mathematical physics (and their generalizations) among all possible equations?

2. Can one find a reasonable (in some sense of this term) boundary value problem for a randomly chosen equation, and if so, how?

3. What is the nature of the pathological phenomena that arise in the case of incorrectly posed boundary value problems?

These questions, and similar ones, need, of course, to be clarified, and are far from having complete answers. Nevertheless, it is clear that they should not be assumed to be merely speculative. The ability to orient one's self in unconventional situations is often valuable for a mathematician or physicist who is concerned with the solution of specific problems. For this reason, the author has tried to make the book accessible to the widest possible circle of readers.

Boundary value problems for partial differential equations constitute a rich and complicated subject, and can be considered from very diverse points of view. The basic approach in this book is through the theory of linear operators in Hilbert space. In certain constructions we also use spaces with other structures, but the Hilbert space of functions of integrable square is fundamental. In this connection, it is frequently most convenient to formulate the solvability properties of a boundary value problem in terms of the properties of the spectrum of an operator associated with the problem.

The first (introductory) chapter "Elements of spectral theory" is a brief exposition of the necessary facts from the corresponding parts of functional analysis.

In the second chapter we discuss general methods of associating a boundary value problem with a linear operator on Hilbert space.

The generality of the questions enumerated above makes it necessary to impose a number of quite stringent restrictions on the operators that we shall study. The elucidation of correct formulations of problems and the study of particular properties of their solutions for "nonclassical" equations is conveniently begun by the consideration of idealized models, for example by considering equations with constant coefficients, with part of the boundary conditions replaced by the condition of periodicity. This allows the application of some version of the method of separation of variables. In essence, the main part of the book (Chapters IV–VI) is based on the use of methods of this kind. By means of these we are led to the consideration of special classes of operator equations for which it is possible to obtain meaningful and rather complete results.

The reader can obtain additional details about the content of the book by looking through it. Numerous general remarks are contained in the introductory subsections, numbered "0".

In conclusion, I offer the following additional remarks. If the books in which the methods of functional analysis are applied to the study of boundary value problems are conditionally divided into two groups:

1) treatises on functional analysis in which differential operators are studied as concrete examples;

2) treatises on the theory of partial differential equations in which functional analysis is one of the methods employed;

then, putting this monograph into the second group, I would emphasize that my intention is that the basic theme should be an exposition of the mechanism of applying the general concepts of functional analysis to the study of definite classes of specific classical entities.

In conclusion, I take this opportunity to thank Professor Sh.A. Alimov for reading the manuscript and making many valuable comments.

A.A. Dezin

# Preface to the English Edition

The main theme of this book is the study of how the solvability of a given linear partial differential equation depends on the choice of the boundary conditions; the principal methods are those of functional analysis. I feel that this theme deserves more attention than it usually receives. Rather than proving many general theorems, I have presented numerous special cases, for which more or less complete results are attainable, in order to illustrate various kinds of results. I hope that these examples will help the reader acquire enough intuition so that they can analyze the particular problems that arise in their own work. For a fuller discussion of the objectives of the book, the reader is referred to the preface to the Russian edition (above).

Shortly after the publication of the first edition, an approach was discovered to many of the problems that are discussed in the main part of the book; it is known as the model-operator method. It has become clear that with this approach one can analyze a large class of diverse problems, both from a unified point of view and in simplified formulations. A number of results in this direction are outlined in an appendix that contains brief summaries, kindly provided by Professor Boas, of some recent papers.

In conclusion, I want to express my gratitude to Professor Boas and to Springer-Verlag for producing this English edition, which should make the book accessible to a wider circle of readers.

Moscow, December 1986 A.A. Dezin

# To the Reader

The book is divided into chapters; the chapters, into sections; the sections, into subsections. Formulas, theorems, and statements are numbered within each section. For a reference within a section, the number is given; for a reference to a different section of the same chapter, also the section (or section and subsection). Otherwise the chapter is also given.

Numbers in square brackets are references to the corresponding books or papers in the bibliography. A reference does not imply that the book or paper cited is the only (or principal) source of the information in question.

The "Halmos symbol" □ marking the end of a proof (possibly only an outline), or to emphasize its absence, is not used altogether systematically. In some cases where no confusion will result, it is omitted.

Definitions are not always set off in separate paragraphs. Frequently they are run into the text. Definitions of concepts are printed in italics.

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