Undergraduate Texts in Mathematics

Springer Science+Business Media, LLC

Undergraduate Texts in Mathematics

Abbott: Understanding Analysis.
Anglin: Mathematics: A Concise History and Philosophy.
Readings in Mathematics.
Anglin/Lambek: The Heritage of Thales.
Readings in Mathematics.
Apostol: Introduction to Analytic Number Theory. Second edition.
Armstrong: Basic Topology.
Armstrong: Groups and Symmetry.
Axler: Linear Algebra Done Right. Second edition.
Beardon: Limits: A New Approach to Real Analysis.
Bak/Newman: Complex Analysis. Second edition.
Banchoff/Wermer: Linear Algebra Through Geometry. Second edition.
Berberian: A First Course in Real Analysis.
Bix: Conics and Cubics: A Concrete Introduction to Algebraic Curves.
Brémaud: An Introduction to Probabilistic Modeling.
Bressoud: Factorization and Primality Testing.
Bressoud: Second Year Calculus.
Readings in Mathematics.
Brickman: Mathematical Introduction to Linear Programming and Game Theory.
Browder: Mathematical Analysis: An Introduction.
Buchmann: Introduction to Cryptography.
Buskes/van Rooij: Topological Spaces: From Distance to Neighborhood.
Callahan: The Geometry of Spacetime: An Introduction to Special and General Relavitity.
Carter/van Brunt: The Lebesgue–Stieltjes Integral: A Practical Introduction.
Cederberg: A Course in Modern Geometries. Second edition.
Childs: A Concrete Introduction to Higher Algebra. Second edition.
Chung: Elementary Probability Theory with Stochastic Processes. Third edition.
Cox/Little/O'Shea: Ideals, Varieties, and Algorithms. Second edition.
Croom: Basic Concepts of Algebraic Topology.
Curtis: Linear Algebra: An Introductory Approach. Fourth edition.
Devlin: The Joy of Sets: Fundamentals of Contemporary Set Theory. Second edition.
Dixmier: General Topology.
Driver: Why Math?
Ebbinghaus/Flum/Thomas: Mathematical Logic. Second edition.
Edgar: Measure, Topology, and Fractal Geometry.
Elaydi: An Introduction to Difference Equations. Second edition.
Exner: An Accompaniment to Higher Mathematics.
Exner: Inside Calculus.
Fine/Rosenberger: The Fundamental Theory of Algebra.
Fischer: Intermediate Real Analysis.
Flanigan/Kazdan: Calculus Two: Linear and Nonlinear Functions. Second edition.
Fleming: Functions of Several Variables. Second edition.
Foulds: Combinatorial Optimization for Undergraduates.
Foulds: Optimization Techniques: An Introduction.
Franklin: Methods of Mathematical Economics.
Frazier: An Introduction to Wavelets Through Linear Algebra.
Gamelin: Complex Analysis.
Gordon: Discrete Probability.
Hairer/Wanner: Analysis by Its History.
Readings in Mathematics.
Halmos: Finite-Dimensional Vector Spaces. Second edition.

(continued after index)

Keith Devlin

The Joy of Sets

Fundamentals of Contemporary Set Theory

Second Edition

With 11 illustrations

Springer

Keith Devlin
School of Science
Saint Mary's College of California
Moraga, CA 94575
USA

Mathematics Subject Classification (2000): 03-01, 03E30, 03E47

Library of Congress Cataloging-in-Publication Data
Devlin, Keith J.
The joy of sets : fundamentals of contemporary set theory / Keith Devlin. -- 2nd ed., completely re-written.
p. cm. -- (Undergraduate texts in mathematics)
Rev. ed. of: Fundamentals of Contemporary set theory / Keith J. Devlin.
June 1992.
Includes bibliographical references and index.
ISBN 978-1-4612-6941-0 ISBN 978-1-4612-0903-4 (eBook)
DOI 10.1007/978-1-4612-0903-4
1. Set theory. I. Devlin, Keith J. Fundamentals of contemporary set theory. II. Title. III. Series.
QA248.038 1993 93-4692
511,3'22--dc20

Printed on acid-free paper.

Originally published by Springer-Verlag New York, Inc in 1993
Softcover reprint of the hardcover 2nd edition 1993
The first edition of this book was published in the Universitext series.

Production managed by Karen Phillips, manufacturing supervised by Vincent Scelta.
Photocomposed pages prepared from the author's LATEX file.

9 8 7 6 5

ISBN 978-1-4612-6941-0 SPIN 10842284

Preface

This book provides an account of those parts of contemporary set theory of direct relevance to other areas of pure mathematics. The intended reader is either an advanced-level mathematics undergraduate, a beginning graduate student in mathematics, or an accomplished mathematician who desires or needs some familiarity with modern set theory. The book is written in a fairly easy-going style, with minimal formalism.

In Chapter 1, the basic principles of set theory are developed in a 'naive' manner. Here the notions of 'set', 'union', 'intersection', 'power set', 'relation', 'function', etc., are defined and discussed. One assumption in writing Chapter 1 has been that, whereas the reader may have met all of these concepts before and be familiar with their usage, she[1] may not have considered the various notions as forming part of the continuous development of a pure subject (namely, set theory). Consequently, the presentation is at the same time rigorous and fast.

Chapter 2 develops the theory of sets proper. Starting with the naive set theory of Chapter 1, I begin by asking the question 'What is a set?' Attempts to give a rigorous answer lead naturally to the axioms of set theory introduced by Zermelo and Fraenkel, which is the system taken as basic in this book. (Zermelo–Fraenkel set theory is in fact the system now accepted in 'contemporary set theory'.) Great emphasis is placed on the evolution of the axioms as 'inevitable' results of an analysis of a highly intuitive notion. For, although set theory has to be developed as an axiomatic theory, occupying as it does a well-established foundational position in mathematics, the axioms themselves must be 'natural'; otherwise everything would reduce to a meaningless game with prescribed rules. After developing the axioms, I go on to discuss the recursion principle—which plays a central role in the development of set theory but is nevertheless still widely misunderstood and rarely appreciated fully—and the Axiom of Choice, where I prove all of the usual variants, such as Zorn's Lemma.

[1] I use both 'he' and 'she' as gender-neutral pronouns interchangeably throughout the book.

Chapter 3 deals with the two basic number systems, the ordinal numbers, and the cardinal numbers. The arithmetics of both systems are developed sufficiently to allow for most applications outside set theory.

In Chapter 4, I delve into the subject set theory itself. Since contemporary set theory is a very large subject, this foray is of necessity very restricted. I have two aims in including it. First, it provides good examples of the previous theory. And second, it gives the reader some idea of the flavor of at least some parts of pure set theory.

Chapter 5 presents a modification of Zermelo–Fraenkel set theory. The Zermelo–Fraenkel system has a major defect as a foundational subject. Many easily formulated problems cannot be solved in the system. The Axiom of Constructibility is an axiom that, when added to the Zermelo–Fraenkel system, eliminates most, if not all, of these undecidable problems.

In Chapter 6, I give an account of the method by which one can *prove* within the Zermelo–Fraenkel system that various statements are themselves *not provable* in that system.

Chapters 5 and 6 are nonrigorous. My aim is to *explain* rather than *develop*. They are included because of their relevance to other areas of mathematics. A detailed investigation of these topics would double the length of this book at the very least and as such is the realm of the set-theorist, though I would, of course, be delighted to think that any of my readers would be encouraged to go further into these matters.

Finally, in Chapter 7, I present an introductory account of an alternative conception of set theory that has proved useful in computer science (and elsewhere), the non-well-founded set theory of Peter Aczel.

Chapters 1 through 3 contain numerous easy exercises. In Chapters 1 and 2, they are formally designated as 'Exercises' and are intended for solution as the reader proceeds. The aim is to provide enough material to help the student understand fully the concepts that are introduced. In Chapter 3, the exercises take the form of simple proofs of basic lemmas, which are left to the reader to provide. Again, the aim is to assist the reader's comprehension.

At the end of each of Chapters 1 through 3, there is also a small selection of problems. These are more challenging than the exercises and constitute digressions from, or extensions of, the main development. In some instances the reader may need to seek assistance in order to do these problems.

This book is a greatly expanded second edition of my earlier *Fundamentals of Contemporary Set Theory*, published by Springer-Verlag in 1979. In addition to the various changes I have made to my original account, I could not resist a change in title, relegating the title of the first edition to a subtitle for the second, thereby enabling me to join the growing ranks of *Joy* books, which began many years ago with *The Joy of Cooking*, achieved worldwide fame, and a certain notoriety, with *The Joy of Sex*, and more recently moved into the mathematical world with *The Joy of* TEX. (This is by no means an exhaustive list.) The new title was suggested to me by my daughter Naomi, herself a college student at the time.

The basis for the first edition was a series of lectures I gave at the University of Bonn, Germany, in the years 1975 and 1976. Chapter 7 is entirely new; its inclusion reflects the changing nature of set theory, as a foundational subject influenced by potential applications. Apart from this addition, the remainder of the account is largely as in the first edition, apart from some stylistic changes and the correction of some minor errors.

I wrote this new edition during the spring of 1992. At that time, I was the Carter Professor of Mathematics at Colby College, in Maine. The manuscript was prepared on an Apple Macintosh IICX computer running the TEXTURES implementation of TEX together with LATEX. I started with an electronic version of the first edition produced during the summer of 1990 by Mehmet Darmar, a Colby mathematics graduate of the Class of 1990, supported by a Colby College faculty assistant summer stipend. Mehmet first created an electronic version of the original book using an optical character reader, and then massaged it into a LATEX document I could work on. The final manuscript was carefully combed for errors by my Colby students Stuart Pitrat and Amy Richters.

I am grateful to Peter Aczel, James Baumgartner, Josep Maria Font, and Carruth McGehee for pointing out a number of minor errors in the first printing of this book.

KEITH DEVLIN

Contents

Preface v

1 Naive Set Theory **1**
- 1.1 What is a Set? . . . 1
- 1.2 Operations on Sets . . . 4
- 1.3 Notation for Sets . . . 6
- 1.4 Sets of Sets . . . 7
- 1.5 Relations . . . 10
- 1.6 Functions . . . 12
- 1.7 Well-Orderings and Ordinals . . . 16
- 1.8 Problems . . . 25

2 The Zermelo–Fraenkel Axioms **29**
- 2.1 The Language of Set Theory . . . 30
- 2.2 The Cumulative Hierarchy of Sets . . . 35
- 2.3 The Zermelo–Fraenkel Axioms . . . 40
- 2.4 Classes . . . 46
- 2.5 Set Theory as an Axiomatic Theory . . . 50
- 2.6 The Recursion Principle . . . 51
- 2.7 The Axiom of Choice . . . 56
- 2.8 Problems . . . 63

3 Ordinal and Cardinal Numbers **66**
- 3.1 Ordinal Numbers . . . 66
- 3.2 Addition of Ordinals . . . 68
- 3.3 Multiplication of Ordinals . . . 69
- 3.4 Sequences of Ordinals . . . 71
- 3.5 Ordinal Exponentiation . . . 74
- 3.6 Cardinality, Cardinal Numbers . . . 75
- 3.7 Arithmetic of Cardinal Numbers . . . 82
- 3.8 Regular and Singular Cardinals . . . 88
- 3.9 Cardinal Exponentiation . . . 91

3.10 Inaccessible Cardinals 95
3.11 Problems 98

4 Topics in Pure Set Theory **101**
4.1 The Borel Hierarchy 101
4.2 Closed Unbounded Sets 103
4.3 Stationary Sets and Regressive Functions 106
4.4 Trees 109
4.5 Extensions of Lebesgue Measure 113
4.6 A Result About the GCH 116

5 The Axiom of Constructibility **120**
5.1 Constructible Sets 120
5.2 The Constructible Hierarchy 123
5.3 The Axiom of Constructibility 124
5.4 The Consistency of $V = L$ 127
5.5 Use of the Axiom of Constructibility 128

6 Independence Proofs in Set Theory **130**
6.1 Some Undecidable Statements 130
6.2 The Idea of a Boolean-Valued Universe 130
6.3 The Boolean-Valued Universe 133
6.4 $V^{\mathcal{B}}$ and V 136
6.5 Boolean-Valued Sets and Independence Proofs 137
6.6 The Nonprovability of the CH 139

7 Non-Well-Founded Set Theory **143**
7.1 Set-Membership Diagrams 145
7.2 The Anti-Foundation Axiom 151
7.3 The Solution Lemma 156
7.4 Inductive Definitions Under AFA 159
7.5 Graphs and Systems 163
7.6 Proof of the Solution Lemma 168
7.7 Co-Inductive Definitions 169
7.8 A Model of ZF^- +AFA 173

Bibliography **185**

Glossary of Symbols **185**

Index **189**