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Georges de Rham

Differentiable Manifolds

Forms, Currents, Harmonic Forms

Translated from the French
by F. R. Smith

Introduction to the English Edition
by S. S. Chern

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Georges de Rham
7, avenue Bergières
CH-1004 Lausanne

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Préface à l'édition anglaise

Je tiens à remercier très sincèrement tous ceux qui ont rendu possible cette traduction de mon livre: d'abord Beno Eckmann, c'est grâce à lui que le projet a été réalisé; Shiing-Shen Chern qui a eu la grande amabilité d'écrire une Introduction; Springer-Verlag pour tous ses efforts et sa coopération; et le traducteur pour son travail consciencieux.

Voici quelques compléments historiques de nature personelle.

Attiré vers les Mathématiques dès 1924, c'est en 1928 et 1929 que j'ai fait ma thèse. En 1930, H. Lebesgue, à qui j'apportais mon manuscrit, m'a dirigé vers Elie Cartan, qui a bien voulu l'examiner et faire un rapport favorable. Ma thèse a paru en juin 1931, précédée de deux Notes aux Comptes Rendus en 1928 et 1929.

Peu après, S. Lefschetz écrit à H. Lebesgue "... vous nous rendriez grand service en suggérant à M. de Rham de nous envoyer quelques exemplaires de sa thèse. M. Hodge nous en a exposé la partie analytique...".

Ce fut, semble-t-il, l'origine des travaux de Hodge publiés vers 1934–1936. L'énoncé de son théorème frappe par sa beauté et sa simplicité. Mais ses démonstrations m'ont paru très pénibles et trop difficiles. Ce qui m'a amené à reprendre le problème dans les travaux publiés en 1946. Pendant la guerre, les "Annals of Mathematics" ne parvenaient pas dans nos bibliothèques et j'ai ignoré le travail de H. Weyl auquel S. S. Chern fait heureusement allusion dans son Introduction. Je n'en ai eu connaissance que plus tard.

En 1950, à l'Institute for Advanced Study de Princeton, à la demande précisément de H. Weyl à qui je dois beaucoup pour l'intérêt qu'il m'a témoigné, j'ai fait une série d'exposés "Harmonic Integrals". De là, grâce à l'intérêt et l'amitié d'André Weil, est issu mon livre.

Lausanne, en mai 1984 — Georges de Rham

Introduction to the English Edition

William Hodge's theory of harmonic integrals was both bold and imaginative. In one step he found the key to the n-dimensional generalization of geometric function theory. His proof of the fundamental theorem contained a serious gap. This was filled in a masterful way by Hermann Weyl, using his earlier results on potential theory.

Professor de Rham's book is an introduction to differentiable manifolds. Its main objective seems to be the first detailed proof, different from Hodge-Weyl, of Hodge's fundamental theorem. It must have given him great pleasure in writing the book, for Hodge theory is a natural culmination of the de Rham theory.

In n-dimensional geometry a fundamental notion is the "duality" between chains and cochains, or domains of integration and the integrands. While the boundary operator is a global operator, the coboundary operator, i.e. exterior differentiation, is local. This makes cohomology theory a more convenient tool for analytical treatment and for applications. Poincaré recognized the importance of the multiple integrals and stated the main "theorems", while Elie Cartan developed the foundations of the exterior differential algebra and applied it to mechanics, differential systems, and differential geometry. The global theory was completed by de Rham's famous thesis in 1931. The thesis was long, because at that time topology was homology theory and the notion of cohomology did not exist.

A notion which includes both chains and cochains is that of a "current". This was introduced by de Rham and used effectively throughout the book. A zero-dimensional current is a distribution (in the sense of Laurent Schwartz), now a fundamental concept in mathematics.

There are now other proofs of Hodge's theorem. Perhaps the most natural approach is through pseudo-differential operators; cf. [5], [6]. The Milgram-Rosenbloom proof using the heat equation method is an idea with broad repercussions [3] Morrey, Eells, and Friedrichs gave a proof using a variational method [4].

Hodge's theorem admits various extensions. The most important one is to cohomology theory with a coefficient sheaf, which was introduced by J. Leray and developed for the complex structure with great success by Henri Cartan and J-P. Serre [1], [6]. Its harmonic theory was first worked out by K. Kodaira [2]. When geometrical information is available, the harmonic theory allows the proof of "vanishing theorems" on cohomology groups, using an idea originated from S. Bochner. Such vanishing theorems are of great importance.

Modern developments in the general area of "elliptic operators on manifolds", such as the index theory and the spectral theory, have raced way beyond the content of this book. I believe, however, that in his enthusiasm for new results a mathematician will be well-advised to stop at this landmark, where he will have a lot to learn both on the mathematics and on the mathematical style.

Berkeley, February 1984 S. S. Chern

References

[1] Griffiths, P., Harris, J.: Principles of Algebraic Geometry. John Wiley 1978
[2] Kodaira, K.: On a differential-geometric method in the theory of analytic stacks. Proc. Nat. Acad. Sci., USA, vol. 29 (1953), 1268–1273
[3] Milgram, A.N., Rosenbloom, P.C.: Harmonic forms and heat conduction, I, II. Proc. Nat. Acad. Sci., USA, vol. 37 (1951), 180–184, 435–438
[4] Morrey, C.B.: Multiple Integrals in the Calculus of Variations. Grundlehren der math. Wiss. 130, Springer 1966
[5] Nirenberg, L.: Pseudo-differential operators, Global Analysis. Proc. Symp. Pure Math., vol. 16, Amer. Math. Soc., 149–167, 1970
[6] Wells, R.O.: Differential Analysis on Complex Manifolds. Prentice Hall, Inc. 1973; second edition, Graduate Texts in Mathematics, no. 65, Springer 1980

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