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Robert Dautray Jacques-Louis Lions

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Evolution Problems I

With the Collaboration of
Michel Artola, Michel Cessenat and Hélène Lanchon

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Translation Editor: Ian N. Sneddon

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Robert Dautray
12 rue du Capitaine Scott
75015 Paris, France

Jacques-Louis Lions
Collège de France
3 rue d'Ulm
75231 Paris Cedex 5, France

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Introduction to Volume 5

This volume and Vol. 6 form the third part of a coherent work which we intend to be useful to engineers, physicists, chemists[1], etc . . . who need a method to solve their stationary or evolutionary problems.

1) These two volumes study *evolution problems*[2], that is to say, problems depending on time. These may, for example, be of the form: find a solution $u(x, t)$ of

$$(P) \qquad \begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + Au = f, \quad x \in \Omega, \quad t > 0 \\ u|_{t=0} = u_0 \quad in\, \Omega \end{cases}$$

with conditions on u on the boundary $\partial\Omega$, $t > 0$.

Volume 5 starts with Chap. XIV, which treats *problems in* $\mathbb{R}^n$ (i.e. problem (P) with $\Omega = \mathbb{R}^n$; that is, *Cauchy problems*). We introduce, at that stage, the types of problems which we shall consider throughout Vol. 3, problems related to the *heat flow equation* (also called the diffusion equation), problems related to the *wave equation* and problems related to the *Schrödinger equation.*

Chapter XV treats problem (P) in $\Omega \subset \mathbb{R}^n$ by *diagonalising the operator A*, that is to say by using the spectral decomposition of the operator A, *which we assume to be self-adjoint.* To this end we use the results of the spectral theory presented in Chap. VIII. The method of diagonalisation (also called the *Fourier* method) leads to an explicit form of the solution in modes, which is very useful in physical or mechanical applications. But these are only calculable if we can perform a *numerical calculation* of the spectral decomposition of A[3].

The method of the *Laplace transform* of problem (P) (with respect to the variable t), which is often used in such applications as electronics, control systems, robotics etc. . . , is treated in Chap. XVI. The method of Laplace transformation also gives us an explicit expression for the solution. But this, as we shall see in Chap. XV, is not always numerically calculable[3].

In Chap. XVII, we show that for a large class of problems, the solution u of problem (P) can be put in the form $u(t) = G(t)u_0$; $\{G(t)\}$ is then a family of operators depending on time t, called a *semigroup* because of the property $G(t + s) = G(t)G(s)$ for all t and $s \geqslant 0$ (we have a group if the same condition

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[1] We refer in the text to these various categories as 'practitioners'.

[2] Recall that Vols. 1 to 4 treated *stationary* problems, that is to say problems independent of time. These are, for example, of the form: find a solution u of $Au = f$ in a domain Ω, A being a differential or integro-differential operator, with conditions on u on the boundary $\partial\Omega$ of Ω.

[3] An expression for the numerically calculated solution will be proposed in Chap. XVIII.

holds for t and s of arbitrary sign). We shall study the relations between the operator A and the family $\{G(t)\}$ which we symbolise by writing:

$$G(t) = e^{tA} .$$

Numerous particular cases of families $\{G(t)\}$ are treated: $\{G(t)\}$ unitary (a propos of which we study *Stone's theorem*), compact, differentiable, holomorphic, contraction, etc. . . These semigroups characterise the modes of evolution that allow us to better understand the properties of the functions $u(t)$ which are solutions of problem (P).

These categories of semigroups are met in the solutions of the three types of problems cited above and allow a profound study of the solutions (always written in an explicit form, but whose numerical expressions may be difficult to calculate[(3)]).

The Trotter formula, which allows us to give a precise meaning to certain limits in semigroups (like the *Feynman-Kac formula*) ends this chapter.

The Laplace transform and semigroup methods *do not assume* the operators are self-adjoint but instead assume the *coefficients are independent of time*. However, we can – particularly with semi-groups – approach problems in which the coefficients depend on time, which is very important, particularly for *nonlinear* problems; we must then make hypotheses, that are technically complicated, on the way in which the *domains* of the operators $A(t)$ (which replace A in the formulation of problem (P)) depend on t. We refer, in particular, to the work of T. Kato.

The simplest and most powerful methods which are *at the same time* applicable to *asymmetric, time-dependent* operators are *variational methods*.

Chapter XVIII *deals with these methods, which allow us to construct the solution u* of problem (P) (by using finite-dimensional spaces). Further, they have, as we have said, the advantage of allowing us to treat problems more general than do the above methods. Besides problems of the above type in their most general form (coefficients depending on x and t), some new problems (such as delay problems) are discussed. This chapter is therefore the centre of Vol. 3, and many of the mathematical tools previously developed lead to it. Additionally the variational methods developed are "a point of departure" for the study of *nonlinear cases.*

2) As before, in the writing of Vol. 5 we have had the benefit of the collaboration of numerous colleagues. We give below the authors of various contributions, chapter by chapter:

Chapter XIV: M. Artola, M. Cessenat
Chapter XV: M. Artola, M. Cessenat, H. Lanchon
Chapter XVI: M. Artola, M. Cessenat
Chapter XVII: M. Artola, M. Cessenat
Chapter XVIII: M. Artola, M. Cessenat.

We also thank P. Bénilan, A. Gervat, R. Glowinski, P. A. Raviart, L. Tartar and R. Temam for reading certain texts, for their advice and for their suggestions.
We extend particular thanks to M. Artola for his essential role in the writing of Chaps. XIV to XVIII.

M. Cessenat has continued, in this volume, his task of careful rereading, accompanied by some very judicious suggestions and propositions. Moreover, he has contributed to Chaps. XIV to XVIII, in particular in the examples, but also in numerous aspects of the exposition of the methods..
We renew our thanks to J. M. Moreau, whose effort is maintained with the same efficiency as in the previous volumes.

3) Our objective, pursued in the course of these volumes, has been the mathematical and numerical study of linear models encountered in the natural and technical sciences, however many analogous models are to be found in the life sciences and economics.
The process of mathematical modelling is complex. It is evolving rapidly, thanks above all to that fantastic tool, the computer (which is clearly still *far* from reaching its limits). Indeed, computers allow us to approximate the mathematical model by sets of equations judged, until now, to be totally intractable. This has, naturally, strongly encouraged practitioners to reconsider, complete, and refine their various models, and make them, little by little, closer to "reality".
In general these lead to *non*linear systems, but one of the most powerful tools for the study of systems of nonlinear partial differential equations is that of *linearisation.* It is therefore *indispensable* to rely on the linear theories presented here. It is obviously not indispensable to know in detail *all* the methods presented here; but faced with a given problem, we must choose a method, and having chosen one, we must be able to follow it without reading all the chapters of the book; we hope that the different adjoining texts – perspectives, directions for the reader, list of equations, table of notation, index – allow the reader to proceed in this way (this has been, we think, achieved, but at the price of repetition and we hope that this will not irritate the reader of several successive chapters).
The ultimate aim is obviously the *understanding* of phenomena, so as to be able to control them, and this understanding comes in three great stages: modelling, starting from fundamental physical principles, mathematical and numerical analysis, computer processing and returning to the physical interpretation. It is in this perspective that the authors have attempted to place themselves

R. Dautray, J.-L. Lions

Practical Guide for the Reader

1. Designation of subdivisions of the text:
number of a chapter: in Roman numerals
number of major division of a chapter: the sign § followed by a numeral
number of section: a numeral following the above
number of a sub-section: a numeral following the above.

Example: II, §3.5.2, denotes chapter II, §3, section 5, subsection 2.

2. *Within each division* (§), the equations, definitions, theorems, propositions, corollaries, lemmas, remarks and examples are *each* numbered consecutively beginning with the number 1.

3. The *table of notations* used is placed at the end of each volume.

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Chapter XV. Evolution Problems: The Method of Diagonalisation

Chapter XVI. Evolution Problems: The Method of the Laplace Transform

Chapter XVII. Evolution Problems: The Method of Semigroups

Chapter XVIII. Evolution Problems: Variational Methods