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for Science and Technology

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Mathematical Analysis and Numerical Methods for Science and Technology

Volume 1
Physical Origins and Classical Methods

With the Collaboration of
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Preface

In the first years of the 1970's Robert Dautray engaged in conversations with Jacques Yvon, High-Commissioner of Atomic energy, of the necessity of publishing mathematical works of the highest level to put at the disposal of the scientific community a synthesis of the modern methods of calculating physical phenomena.

It is necessary to get away from the habit of treating mathematical concepts as elegant abstract entities little used in practice. We must develop a technique, but without falling into an impoverishing utilitarianism. The competence of the Commissariat à l'Energie Atomique in this matter can provide a support of exceptional value for such an enterprise.

The work which I have the pleasure to present realises the synthesis of mathematical methods, seen from the angle of their applications, and of use in designing computer programs. It should be seen as complete as possible for the present moment, with the present degree of development of each of the subjects. It is this specific approach which creates the richness of this work, at the same time a considerable achievement and a harbinger of the future. The encounter to which it gives rise among the originators of mathematical thought, the users of these concepts and computer scientists will be fruitful for the solution of the great problems which remain to be treated, should they arise from the mathematical structure itself (for example from non-linearities) or from the architecture of computers, such as parallel computers.

This task has led to planning, spread over ten consecutive years of strenous work, by two exceptional men – the physicist Robert Dautray and the mathematician Jacques-Louis Lions. In addition, they have enlisted the assistance of younger research workers, so it is fair to include them in our thanks for a work which, deemed indispensible thoughout, does not seem to me to have been undertaken quite at this level anywhere else in the world.

Jean Teillac
High Commissioner of Atomic Energy

General Introduction

1. A very great number of the problems of mathematical physics can be "modelled" by partial differential equations. By a "model", we mean a set of equations (or inequations) which, together with boundary conditions (expressed on the boundary of the spatial domain where the phenomenon is studied) and, when the phenomenon is evolutionary, with initial conditions, allow us to define the state of the system. This is also called modelling by "distributed systems".

Naturally the description of the model (or of a model, since the same phenomenon can often be described, in conditions not always strictly equivalent, by different state variables) is an important – but not decisive – step.

Further, we must "study" the model, i.e. deduce qualitative or quantitative properties which

(a) recover, in simple conditions, observations (measurements) already made.

(b) give supplentary information about the system.

It has been observed for a long time that the majority of the phenomena of mathematical physics are *non-linear,* among the most celebrated cases being Boltzmann's equation in statistical mechanics, the Navier-Stokes in fluid mechanics (equations which moreover constitute an *approximation* to Boltzmann's equation) von Karman's equations governing the large displacements of flat plates, etc.

However, having the possibility of using in a systematic – and almost "commonplace" – way the procedures for calculating *approximate solutions* of the state of the system, precise results can generally only be obtained in *the linear cases.*

Certain physical problems can be modelled *directly* (i.e. without *approximations*) by linear equations: this is notably the case of the equation of transport of neutrons. Other phenomena can be deduced from "truly" non-linear systems by neglecting certain terms (which is valid in certain situations: "small" displacements, "slow" motions ...) produced by linearisation about a particular solution.

As, in addition, the *methods* brought into play for the solution of linear problems play an essential role in all the non-linear situations known to this day, it is indispensible to begin with the study of *linear distributed models,* or again with *boundary value problems for linear partial differential equations* (with brief incursions into the domain of *linear* integral equations, equations which we can deduce from linear partial differential equations, or can appear directly in this form).

It is the aim of this work to study linear distributed models, completely concentrating in particular on *physical examples* (from various sources), by *the general*

methods of linear analysis (stating very clearly the application of these methods to physically important situations).
We have tried to render the material accessible to a reader the level of whose knowledge is pretty nearly that of the Lebesgue integral; the (indispensible) theory of distributions is recalled in the Appendix to Volume 2.

2. The theory of partial differential equations constitutes today one of the important topics of scientific understanding.
The principal reasons for this state of affairs are, on the one hand the progress of *mathematical analysis* and, on the other hand, the arrival of *the technique of numerical calculus* which remained, for partial differential equations, almost totally inadequate until the 1950's. In effect, the arrival of computers, and their immense and unceasing progress, have allowed us – for the first time in history – to *calculate, beginning with the models,* quantities which, formerly we were able only to estimate very approximately and, perhaps over all, to calculate them *accurately* and rapidly, and hence the (fundamental) possibility for research workers and engineers to be able to use the numerical results for the modification or adaptation of scientific arguments, of experiments or of constructions in progress.
All that explains why, in very differing subjects, *modelling by partial differential equations,* followed by theoretical analysis, then numerical analysis, and then in its turn with comparison with experiment has become a *basic method of procedure.* Every aspect of technical and industrial activity is concerned; this procedure is indispensible in the preparation or experiments and of trials and their interpretation, technical studies, the development of manufacturing processes, maintenance, reliability, etc. ... Thus:
Modern equipment has to operate in high performance with certain materials. In the 1950's, the calculation of the strength of materials was carried out with high safety factors (for example, 5 or more) on the stress experienced by the material of a given piece at a given point. Today, when we calculate a stress with precision, the safety factor which we take is of the order of 1.4 or less (for example in aeronautical, nuclear, automobile engineering etc. ...) and then in the very best conditions for users.
Similarly, the reliability and security demanded by many modern techniques, from nuclear engineering to aeronautics, from aerospace to large public works (high speed rail transport, highway construction, generation and distribution of electricity, etc. ...) require the accumulation both of the safety factors and, as well require that each of the details is studied and is represented with great precision. No element is any longer "neglected", then and only then the faithful mathematical representation allows us to examine closely the least detail and underline the predictions.
Modelling by distributed systems has become similarly the basis of many disciplines in physics (plasmas, new materials, etc. ...) in the space and earth sciences (astrophysics, geophysics etc. ...), in chemistry and obviously in all branches of mechanics (a number of which have already been cited above).
Without wishing to draw up here an exhaustive list, we should add that, by the intervention, notably of dynamic programming, (non-linear) partial differen-

tial equations play an important role in the *management sciences* (stocks, energy, etc. . . .).
Distributed models are similarly involved, and more and more, in the *life sciences*.

3. Plan of the Work. We give here a general sketch of the content of the chapters grouped by volumes. Each volume begins with a slightly more detailed account of its contents.
We begin by giving in Chapter I a list of mathematical models, important in applications to physics and to the mechanics of continuous media which can lead to linear problems.
The study of *stationary linear problems* begins with a review, in Chapter II, of the possibilities of making use of *classical methods*. We discover that their limits are quickly reached. We examine in Chapter III the possibilities of applying *functional transformations* (Fourier series, the Fourier, Mellin and Hankel transforms etc. . . .). We touch there similarly on the limitations on their application. These limitations show the usefulness of working on sets (of distributions) very much more "extensive" than the sets of continuous functions considered in Chapter II: these are the spaces of general distributions introduced in Chapter III and the *Sobolev spaces* studied in Chapter IV.
The study of *differential operators* in the spaces of general distributions allows us to distinguish the properties of these operators (elliptic, parabolic and hyperbolic operators; local character of mathematical models using differential operators; characteristics etc. . . .) which will serve us well throughout this work; this is the subject matter of Chapter V.
Throughout the whole of this work we shall have to handle *operators;* "operations" on these operators and their approximations are explained, in Chapter VI, in the mathematical situations used in the present work.
The mathematical techniques thus gathered together allow us to treat *variational methods,* which make up the subject of Chapter VII and whose potential for application extends to many non-linear problems.
Numerous *spectral* problems arise in applications (calculation of energy levels and states in quantum mechanics, critical conditions in neutronics, transmission in a wave guide or in an optical fibre etc. . . .). The spectral theory which enables us to treat such problems is seen in Chapter VIII within the perspective of typical applications; it includes especially the study of the continuous spectrum, source of many difficulties. Examples of applications are give in Chapter IX.
A problem which is elliptic or hyperbolic according to the value of a parameter is treated in Chapter X: *Tricomi's problem* (in fluid mechanics it corresponds to the passage from subsonic flight to supersonic flight).
Mathematical models involving integrals permit the representation of actions at a distance (in physical space, electric potential; in time, the memory of a viscoelastic body; in the space of velocities, change of velocity as a result of collisions). *Integral equations* which come into play require the methods treated in Chapter IX.
Finally, the *numerical methods* to treat stationary problems form the subject matter of Chapters XII and XIII.

Linear evolution problems are treated first of all in the whole physical space in Chapter XIV.
The *diagonalisation* method, using the spectral theory of operators, which is the basis of several practical methods (giving rise to the decomposition into modes), is treated in Chapter XV. The method of the *Laplace transform* can be used to treat numerous evolution problems; it is considered in Chapter XVI.
The solution $u(t)$ of a large class of evolution problems can be written in the form $u(t) = G(t)\,u_0$, where u_0 is the initial value of u and u_0 is the initial value of u and $G(t)$ a family of operators forming a *semi-group*. The types of evolution of solutions can then be an examination of the various families of semi-groups $G(t)$. This provides a method, in certain ways more general than the preceding ones, to treat evolution problems. This is the subject of Chapter XVII.
Finally, the constructive methods of solving evolution problems (using constructions of solution in finite-dimensional spaces), the *variational methods*, are seen in Chapter XVIII [1]. The *Navier-Stokes problem* (in the linearised case) requires particular variational methods. These are described in Chapter XIX.
Chapter XX presents the *numerical methods* for linear evolution problems.
The problems involving a *transport equation* are not included in the categories treated in Chapters XIV to XVIII, since they take into account the very particular type of properties of the transport operator (transport of neutrons, transport of molecules and Boltzmann's equation, transport of charged particles and Vlasov's equation). A special chapter, Chapter XXI, is therefore devoted to these problems.
Later chapters study other aspects of certain of the problems studied in the present work (relations between problems of partial differential equations and probabilities, propagation of waves, etc. ...).

4. The writing of this work has been conceived with the object of making it accessible to an engineer or to an aspiring research worker taking only the information he needs to treat his problem; a restricted reading is therefore possible if the reader is guided by the index, the table of contents and the table of notations.

5. In producing this work the undersigned have benifitted from the collaboration of many colleagues: Michel Artola, Marc Authier, Claude Bardos, Philippe Bénilan, Michel Bernadou, Michel Cessenat, Jean-Michel Combes, André Gervat, Alain Kavenoky, Hélène Lanchon, Patrick Lascaux, Bertrand Mercier, Jean-Claude Nédélec, Olivier Pironneau, Jacques Planchard, Bruno Scheurer, Claude Wild, Claude Zuily.
Their contributions and the contributions on specific points due to several other colleagues will be acknowledged at the beginning of each volume.
The manuscript was read with particular care by Michel Cessenat whom we thank most warmly. Considering the size and the diversity of this work, the task he performed is considerable. In addition, Michel Cessenat proposed complementary or corrected texts, valuable contributions which we have often retained.

[1] These methods can be similarly extended to non-linear problems.

Our thanks go similarly to Jean-Marie Moreau for his important work in compiling the bibliographies for each volume, for reading the text and bringing it to the point of publication.
This work would not have seen the light of day without the support of the Atomic Energy Commission (C.E.A.): Jacques Yvon, then High Commissioner of Atomic Energy accepted our proposition immediately, as he could foresee its future development. He made its publication one of the scientific enterprises of the C.E.A. Our respective experiences had, in effect, as early as the end of the 1960's, confirmed our belief in the importance of the existence of a work of reference of this type. By the beginning of 1970's, we had elaborated our ideas into a plan, taking account of the needs of engineers, physicists and workers in mechanics etc. ... Jacques Yvon together with ourselves, wished to spread and put within their grasp the abundant recent work of mathematicians and numerical analysts.
In the initial period, at the time of preliminary drafts and launching the project, we benifitted from the initiative of Robert Lattes, who was then Scientific Adviser of the C.E.A.
We are grateful to Paul Bonnet, Inspector General of the C.E.A. for having inaugurated the C.E.A. collection with this work.
We have greatly valued, and are immensely grateful for, the initial help and encouragement of Jules Horowitz, Director at the C.E.A. who with his great experience in mathematical physics showed an immediate understanding of our aims.
Nothing would have been achieved in reaching the final result without the clear and active understanding of Jacques Chevallier, Director at the C.E.A.
We thank here also Michel Pecquer and Gérard Renon, Administrator General, as well as Jean Teillac, High Commissioner, of the C.E.A. whose constant and manifest approval, personally expressed, has been a source of permanent encouragement.

R. Dautray, J.-L. Lions

Practical Guide for the Reader

(1) Designation of the subdivisions of the text:
Number of chapters: in roman numerals;
Number of major divisions: the sign § followed by a numeral;
Number of sections: a numeral following the preceding;
Number of sub-sections: a numeral following the preceding;
etc. ...
For example: II, § 3.5.2, denotes Chapter II, § 3, section 5, subsection 2.
(2) *In the interior of each division* (§), the equations, definitions, theorems, propositions, corollaries, lemmas, remarks and examples are numbered separately in sequence beginning with the *numeral 1*.
(3) The *table of the notations used* appears at the end of each volume.

Introduction to Volume 1

Chapter I gives the *principal physical examples* studied in this work (these examples come from physics, from mechanics, from chemistry, etc. . . .). A first (rudimentary) attempt at the classification of the problems is made.
In *all* the phenomena modelled by partial differential equations, and for reasons that are given in the text, a very important role is played by the *Laplacian operator*

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

in rectangular coordinates: This is why Chapter II is devoted to a direct study of the *principal questions linked with this operator,* "direct" *signifying here: without the use of techniques other than those of classical analysis.*
We give below the authors of various contributions to these two chapters.

Chapter I: H. Lanchon, M. Cessenat, A. Gervat, A. Kavenoky.
Chapter II: P. Bénilan, sole author of this Chapter.

We similarly wish to thank R. Balian, C. Bardos, A. Bossavit, C. Cohen-Tannoudji, G. Fournet, A. Kavenoky and E. Roubine for reading certain portions of the text and for their advice on modifying them.

The reader wishing to acquaint himself rapidly with the essential mathematical and numerical methods should be able to make use of this volume and the subsequent Vol. 2 by leaving for a later, deeper study §§ 5–8 of Chapter II of this volume and §§ 4, 5 of Chapter V of Vol. 2. These divisions are denoted by an asterisk * placed at their beginning, an asterisk which, moreover, appears in the table of contents.
We have placed the table of notations at the end of this volume.

R. Dautray, J.-L. Lions

Table of Contents

Chapter II. The Laplace Operator