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Unsolved Problems in Intuitive Mathematics
Volume II

Hallard T. Croft Kenneth J. Falconer
Richard K. Guy

Unsolved Problems in Geometry

With 66 Figures

Springer Science+
Business Media, LLC

Hallard T. Croft
Peterhouse,
Cambridge, CB2 1RD
England

Richard K. Guy
Department of Mathematics and Statistics
The University of Calgary
Calgary, Alberta T2N 1N4
Canada

Kenneth J. Falconer
School of Mathematics
Bristol University
Bristol, BS8 1TW
England

Series Editor:
Paul R. Halmos
Department of Mathematics
Santa Clara University
Santa Clara, CA 95053
USA

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Preface

For centuries, mathematicians and non-mathematicians alike have been fascinated by geometrical problems, particularly problems that are "intuitive," in the sense of being easy to state, perhaps with the aid of a simple diagram. Sometimes there is an equally simple but ingenious solution; more often any serious attempt requires sophisticated mathematical ideas and techniques. The geometrical problems in this collection are, to the best of our knowledge, unsolved. Some may not be particularly difficult, needing little more than patient calculation. Others may require a clever idea, perhaps relating the problem to another area of mathematics, or invoking an unexpected technique. Some of the problems are very hard indeed, having defeated many well-known mathematicians.

It is hoped that this book will be appreciated at several levels. The research mathematician will find a supply of problems to think about, and should he or she decide to make a serious attempt to solve some of the them, enough references to discover what is already known about the problem. By becoming aware of the state of knowledge on certain topics, mathematicians may notice links with their own special interest, leading to progress on the problem or on their own work. More generally, the book may provide stimulus as "background reading" for mathematicians or others who wish to keep up to date with the state of the art of geometrical problems.

For the interested layman, the book will give an idea of some of the problems that are being tackled by mathematicians today. This may lead to having a go at some of the problems, discovering the difficulties, and perhaps producing a solution, either fallacious or, hopefully, valid. In the area of tiling, for example (see Chapter C), patience and, sometimes, the use of a home computer, have led amateur mathematicians to impressive results.

Perhaps this book will convince others that some of the ideas of mathematics are not wholly beyond their reach, in that the problems, at least, can be understood. This may well lead to the question, "Why are mathematicians wasting their time on such esoteric problems?" One justification might be that, even if the problems themselves have no direct application, the mathematics developed while trying to solve them can be very useful. For example, attempts at proving the (rather useless) Four Color Conjecture led to the development of Graph Theory, a subject of great practical application.

In many developed countries the number of young people embarking on careers of a mathematical nature is dwindling; it is hoped that books such as this may do a little to show them that mathematics *is* a fascinating subject.

Each section in the book describes a problem or a group of related problems. Usually the problems are capable of generalization or variation in many directions; hopefully the adept reader will think of such variations. Frequently, problems are posed in the plane, but could equally well be asked in 3- (or higher-) dimensional space (often resulting in a harder problem). Alternatively, a question about convex sets, say, might be asked about the *special* case of centro-symmetric convex sets or perhaps in the more *general* situation of completely arbitrary sets, resulting in a problem of an entirely different nature.

For convenience, the problems have been divided into seven chapters. However, this division and the arrangement within the chapters is to some extent arbitrary. The many interrelationships between the problems make a completely natural ordering impossible.

References for each group of problems are collected at the end of the section. A few, indicated by square brackets [ ], refer to the list of Standard References at the front of the book; this is done to avoid excessive repetition of certain works. Where useful, we have included the number of the review in Mathematical Reviews, prefixed by MR. Some sections have few references, others a large number. Not all problems have a complete bibliography—a full list of articles that relate to Sylvester's Problem (F12), for example, would fill the book. However, we have done our best to include those books and papers likely to be most helpful to anyone wishing to find out more about a problem, including the most important and recent references and survey articles. Obviously, these will, in turn, lead to further references. Some books and papers listed are not referred to directly in the text but are nonetheless relevant. There are inevitably some omissions, for which we apologize.

This book has a long history, and many correspondents will have despaired of its ever appearing. Early drafts date back to about 1960 when HTC started to collect problems that were unsolved but easy to state, particularly problems on geometry, number theory, and analysis. As a frequent visitor to Cambridge, RKG became aware of the collection and started contributing to it. Interest was reinforced at the East Lansing Geometry Conference in March 1966, in which HTC, RKG, John Conway, Paul Erdős, and Leo Moser participated, and where Moser circulated his "Fifty poorly posed problems in combinatorial geometry." Encouraged by HTC, Erdős, and Moser, RKG started to formalize these lists into a book. The mass of material became far too large for a single volume, with the Number Theory "chapter" resembling a book in its own right. This was published in 1981 as "Unsolved Problems in Number Theory," Volume I of this series.

The geometry and analysis drafts were regularly updated by HTC until about 1970, when the mass of material collected was enormous. However, this part of the project became dormant until KJF, a research student of HTC, became involved in 1986. By this time, progress had been made on the solution of many of the problems, and it was left to KJF with some help from RKG to update and rewrite completely most of the sections and to add further problems to the collection.

There are still many problems that we have collected which would be nice to share. We have in mind (should we live long enough) Volume III on Combinatorics, Graphs, and Games and Volume IV on Analysis.

We are most grateful to the many people who have, over the past 30 years, corresponded and sent us problems, looked at parts of drafts, and made helpful comments. These include G. Blind, D. Chakerian, John Conway, H. S. M. Coxeter, Roy Davies, P. Erdős, L. Fejes Tóth, Z. Füredi, R. J. Gardner, Martin Gardner, Ron Graham, B. Grünbaum, L. M. Kelly, V. Klee, I. Leader, L. Mirsky, Leo Moser, Willy Moser, C. M. Petty, R. Rado, C. A. Rogers, L. A. Santaló, Jonathan Schaer, John Selfridge, H. Steinhaus, G. Wengerodt, and J. Zaks. The bulk of the technical typing was done by Tara Cox and Louise Guy. The figures were drawn by KJF using a combination of a computer graphics package and freehand. The staff at Springer-Verlag in New York have been courteous, competent, and helpful.

In spite of this help, many errors and omissions remain. Some of the problems may have solutions that are unpublished or in papers that we have overlooked, others will doubtless have been solved since going to press. There must be many references that we are unaware of. The history of some of the problems may have become forgotten or confused—it is often unclear who first thought of a particular problem, and many problems undoubtedly occur to several people at about the same time. This book will no doubt perpetuate such confusion. We can do no more than offer our reluctant apologies for all this. We would be glad to hear of any omissions or corrections from readers, so that any future revision may be more accurate. Please send such comments to KJF.

Cambridge, Bristol and Calgary, September 1990 HTC, KJF, RKG

Other Problem Collections

Here are some good sources of general geometrical problems, many of which are lists compiled at conference meetings. Collections of problems on more specific subjects are mentioned in the chapter introductions.

P. Erdős, Some combinatorial problems in geometry, Lecture Notes in Mathematics, 792, Springer-Verlag, New York, 1980, 46–53; *MR* **82d**:51002.
W. Fenchel (ed.), Problems, [Fen] 308–325.
P. M. Gruber & R. Schneider, *Problems in Geometric Convexity*, [TW] 255–278.
R. K. Guy (ed.), Problems, in *The Geometry of Metric and Linear Spaces*, Lecture Notes in Mathematics 490, Springer-Verlag, New York, 1974, 233–244.
D. C. Kay & M. Breen (eds.), [KB] 229–234.
V. L. Klee (ed.), Unsolved problems, [Kle] 495–500.
V. L. Klee, Some unsolved problems in plane geometry, *Math. Mag.* **52** (1979) 131–145; *MR* **80m**:52006.
D. G. Larman & C. A. Rogers (eds.), Problems, Durham Symposium on relations between finite and infinite dimensional convexity, *Bull. London Math. Soc.* **8** (1976) 28–33.
J. E. Littlewood, *Some Problems in Real and Complex Analysis*, Heath, Boston, 1968; *MR* **39** #5777.
R. D. Mauldin (ed.), *The Scottish Book*, [Mau].
Z. A. Melzak, Problems conerned with convexity, *Canad. Math. Bull.* **8** (1965) 565–573; *MR* **33** #4781.
Z. A. Melzak, More problems concerned with convexity, *Canad. Math. Bull.* **11** (1968) 482–494; *MR* **38** #3767.
H. Meschkowski, *Unsolved and Unsolvable Problems in Geometry*, [Mes].
W. Moser & J. Pach, *Research Problems in Discrete Geometry*, [MP].
C. S. Ogilvy, *Tomorow's Math*, Oxford University Press, Oxford, 1962.
H. Steinhaus, *One Hundred Problems in Elementary Mathematics*, Pergamon, Oxford, 1964; *MR* **28** #1110.
S. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960; *MR* **22** #10884.

Journals that publish problems on a regular basis include: *American Mathematical Monthly*, *Colloquium Mathematicum*, *Elemente der Mathematik*, *Mathematical Intelligencer*, *Mathematics Magazine*, *SIAM Review*, and *Crux Mathematicarum.*

Standard References

The following books and conference proceedings are referred to so frequently that we list them here, and refer to them by letters in square brackets throughout the book.

[Bla] W. Blaschke, *Kreis und Kugel*, 1916; Chelsea reprint, New York 1949; *MR* **17**, 887.

[BonFe] T. Bonnesen & W. Fenchel, *Theorie der Konvexer Körper*, Springer, Berlin, 1934; Chelsea reprint, New York 1971; *MR* **51** #8954.

[BF] K. Böröczky & G. Fejes Tóth (eds.), *Intuitive Geometry, Siófok, 1985*, North-Holland, Amsterdam, 1987; *MR* **88h**:52002.

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[DGK] L. Danzer, B. Grünbaum & V. Klee, *Helly's Theorem and its Relatives, in Convexity* (V. L. Klee, ed.), Proc. Symp. Pure Math. 7, Amer. Math. Soc., Providence, 1963, 101–180; *MR* **28** #524.

[DGS] C. Davis, B. Grünbaum & F. A. Sherk (eds.), *The Geometric Vein—The Coxeter Festschrift*, Springer-Verlag, New York, 1981; *MR* **83e**:51003.

[Fej] L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und in Raum*, 2nd ed., Springer, Berlin, 1972; *MR* **50** #5603.

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[Fen] W. Fenchel (ed.), Proceedings of the Colloquium on Convexity, 1965, Københavns Univ. Mat. Inst., 1967; *MR* **35** #5271.

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[GW] P. M. Gruber & J. M. Wills (eds.), *Convexity and its Applications*, Wien 1981, Siegen, 1982, Birkhäuser, Basel, 1983; *MR* **84m**:52001.

[Had] H. Hadwiger, *Altes und Neues über Konvexer Körper*, Birkhäuser, Basel, 1955; *MR* **17**, 401.
[Ham] J. Hammer, *Unsolved Problems Concerning Lattice Points*, Pitman, London, 1977; *MR* **56** #16515.
[HDK] H. Hadwiger & H. Debrunner (translated by V. Klee), *Combinatorial Geometry in the Plane*, Holt, Rinehart, and Winston, New York, 1964; *MR* **29** #1577.
[Kay] D. C. Kay (ed.), Proc. Conf. on Convexity and Combinatorial Geometry, Norman, Oklahoma, 1971, University of Oklahoma, 1971; *MR* **48** #9547.
[KB] D. C. Kay & M. Breen (eds.), *Conference on Convexity and Related Combinatorial Geometry, Oklahoma, 1980*, Marcel Dekker, New York, 1982; *MR* **83b** #52002.
[Kla] D. A. Klarner (ed.), *The Mathematical Gardner*, Wadsworth, Belmont, 1981; *MR* **82b**:00003.
[Kle] V. L. Klee (ed.), Convexity, Proc. Symp. Pure Math. 7, Amer. Math. Soc., Providence, 1963.
[Mau] R. D. Mauldin (ed.), *The Scottish Book*, Birkhäuser, Basel, 1981; *MR* **84m**:00015.
[Mes] H. Meschkowski, *Unsolved and Unsolvable Problems in Geometry*, Oliver and Boyd, 1966; *MR* **35** #7206.
[MP] W. Moser & J. Pach, Research problems in discrete geometry, Mineographed Notes, 1985.
[Rog] C. A. Rogers, *Packing and Covering*, Cambridge University Press, Cambridge, 1964; *MR* **30** #2405.
[RZ] M. Rosenfeld & J. Zaks (eds.), *Proceedings of Jerusalem Conference, 1981*, Ann. Discrete Math. 20, North-Holland, Amsterdam, 1984; *MR* **86g**:52001.
[TW] J. Tölke & J. M. Wills (eds.), *Contributions to Geometry*, Birkhäuser, Basel, 1979; *MR* **81b**:52002.

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