

Richard Courant • Fritz John

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Introduction to Calculus and Analysis

Volume I

With 204 Illustrations



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