

Richard Courant • Fritz John

# Introduction to Calculus and Analysis

Volume II/1  
Chapters 1-4

Reprint of the 1989 Edition



Springer

---

Originally published in 1974 by Interscience Publishers, a division  
of John Wiley and Sons, Inc.  
Reprinted in 1989 by Springer-Verlag New York, Inc.

---

Mathematics Subject Classification (1991): 26xx, 26-01

Cataloging-in-Publication Data applied for

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Courant, Richard:

Introduction to calculus and analysis / Richard Courant; Fritz John.- Reprint.- Berlin; Heidelberg;  
New York; Barcelona; Hong Kong; London; Milan; Paris; Singapore; Tokyo: Springer  
(Classics in mathematics)

Vol.2. / With the assistance of Albert A. Blank and Alan Solomon 1. Chapter 1-4.- Reprint of  
the 1989 ed.- 2000

ISBN 978-3-540-66569-4 ISBN 978-3-642-57149-7 (eBook)

DOI 10.1007/978-3-642-57149-7

Photograph of Richard Courant from: C. Reid, *Courant in Göttingen  
and New York. The Story of an Improbable Mathematician*,  
Springer New York, 1976

Photograph of Fritz John by kind permission of The Courant Institute  
of Mathematical Sciences, New York

ISSN 1431-0821

ISBN 978-3-540-66569-4

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 2000

Originally published by Springer-Verlag Berlin Heidelberg New York in 2000

The use of general descriptive names, registered names, trademarks etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

SPIN 10991329 41/3111CK - 5 4 3 2 - Printed on acid-free paper

Richard Courant   Fritz John

# Introduction to Calculus and Analysis

Volume II

With the assistance of  
Albert A. Blank and Alan Solomon

With 120 Illustrations



Springer

Richard Courant (1888 - 1972)                      Fritz John  
Courant Institute of Mathematical Sciences  
New York University  
New York, NY 10012

Originally published in 1974 by Interscience Publishers, a division of John Wiley and Sons, Inc.

---

Mathematical Subject Classification: 26xx, 26-01

---

Printed on acid-free paper.

Copyright 1989 Springer Science+Business Media New York  
Originally published by Springer-Verlag New York, Inc in 1989

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher Springer Science+Business Media, LLC, except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Act, may accordingly be used freely by anyone.

9 8 7 6 5 4 3

ISBN 978-3-540-66569-4            ISBN 978-3-642-57149-7 (eBook)  
DOI 10.1007/978-3-642-57149-7

SPIN 10691322

# *Contents*

## *Chapter 1 Functions of Several Variables and Their Derivatives*

<b>1.1 Points and Points Sets in the Plane and in Space</b>	<b>1</b>
a. Sequences of points. Convergence, 1	
b. Sets of points in the plane, 3	
c. The boundary of a set. Closed and open sets, 6	
d. Closure as set of limit points, 9	
e. Points and sets of points in space, 9	
<b>1.2 Functions of Several Independent Variables</b>	<b>11</b>
a. Functions and their domains, 11	
b. The simplest types of functions, 12	
c. Geometrical representation of functions, 13	
<b>1.3 Continuity</b>	<b>17</b>
a. Definition, 17	
b. The concept of limit of a function of several variables, 19	
c. The order to which a function vanishes, 22	
<b>1.4 The Partial Derivatives of a Function</b>	<b>26</b>
a. Definition. Geometrical representation, 26	
b. Examples, 32	
c. Continuity and the existence of partial derivatives, 34	

	d. Change of the order of differentiation, 36	
<b>1.5</b>	<b>The Differential of a Function and Its Geometrical Meaning</b>	<b>40</b>
	a. The concept of differentiability, 40 b. Directional derivatives, 43 c. Geometric interpretation of differentiability, The tangent plane, 46 d. The total differential of a function, 49 e. Application to the calculus of errors, 52	
<b>1.6</b>	<b>Functions of Functions (Compound Functions) and the Introduction of New Independent Variables</b>	<b>53</b>
	a. Compound functions. The chain rule, 53 b. Examples, 59 c. Change of independent variables, 60	
<b>1.7</b>	<b>The Mean Value Theorem and Taylor's Theorem for Functions of Several Variables</b>	<b>64</b>
	a. Preliminary remarks about approximation by polynomials, 64 b. The mean value theorem, 66 c. Taylor's theorem for several independent variables, 68	
<b>1.8</b>	<b>Integrals of a Function Depending on a Parameter</b>	<b>71</b>
	a. Examples and definitions, 71 b. Continuity and differentiability of an integral with respect to the parameter, 74 c. Interchange of integrations. Smoothing of functions, 80	
<b>1.9</b>	<b>Differentials and Line Integrals</b>	<b>82</b>
	a. Linear differential forms, 82	

b. Line integrals of linear differential forms, 85 c. Dependence of line integrals on endpoints, 92

<b>1.10 The Fundamental Theorem on Integrability of Linear Differential Forms</b>	<b>95</b>
a. Integration of total differentials, 95	
b. Necessary conditions for line integrals to depend only on the end points, 96	
c. Insufficiency of the integrability conditions, 98	
d. Simply connected sets, 102	
e. The fundamental theorem, 104	

**APPENDIX**

<b>A.1. The Principle of the Point of Accumulation in Several Dimensions and Its Applications</b>	<b>107</b>
a. The principle of the point of accumulation, 107	
b. Cauchy's convergence test. Compactness, 108	
c. The Heine-Borel covering theorem, 109	
d. An application of the Heine-Borel theorem to closed sets contains in open sets, 110.	
<b>A.2. Basic Properties of Continuous Functions</b>	<b>112</b>
<b>A.3. Basic Notions of the Theory of Point Sets</b>	<b>113</b>
a. Sets and sub-sets, 113	
b. Union and intersection of sets, 115	
c. Applications to sets of points in the plane, 117.	
<b>A.4. Homogeneous functions.</b>	<b>119</b>

## **Chapter 2** *Vectors, Matrices, Linear Transformations*

<b>2.1</b>	<b>Operations with Vectors</b>	<b>122</b>
	a. Definition of vectors, 122	
	b. Geometric representation of vectors, 124	
	c. Length of vectors. Angles between directions, 127	
	d. Scalar products of vectors, 131	
	e. Equation of hyperplanes in vector form, 133	
	f. Linear dependence of vectors and systems of linear equations, 136	
<b>2.2</b>	<b>Matrices and Linear Transformations</b>	<b>143</b>
	a. Change of base. Linear spaces, 143	
	b. Matrices, 146	
	c. Operations with matrices, 150	
	d. Square matrices. The reciprocal of a matrix. Orthogonal matrices. 153	
<b>2.3</b>	<b>Determinants</b>	<b>159</b>
	a. Determinants of second and third order, 159	
	b. Linear and multilinear forms of vectors, 163	
	c. Alternating multilinear forms. Definition of determinants, 166	
	d. Principal properties of determinants, 171	
	e. Application of determinants to systems of linear equations. 175	
<b>2.4</b>	<b>Geometrical Interpretation of Determinants</b>	<b>180</b>
	a. Vector products and volumes of parallelepipeds in three-dimensional space, 180	
	b. Expansion of a determinant with respect to a column. Vector products in higher dimensions, 187	
	c. Areas of parallelograms and volumes of parallelepipeds in	



higher dimensions, 190 d. Orientation of parallelepipeds in  $n$ -dimensional space, 195 e. Orientation of planes and hyperplanes, 200 f. Change of volume of parallelepipeds in linear transformations, 201

- 2.5 Vector Notions in Analysis** **204**  
 a. Vector fields, 204 b. Gradient of a scalar, 205 c. Divergence and curl of a vector field, 208 d. Families of vectors. Application to the theory of curves in space and to motion of particles, 211

## *Chapter 3 Developments and Applications of the Differential Calculus*

- 3.1 Implicit Functions** **218**  
 a. General remarks, 218 b. Geometrical interpretation, 219 c. The implicit function theorem, 221 d. Proof of the implicit function theorem, 225 e. The implicit function theorem for more than two independent variables, 228
- 3.2 Curves and Surfaces in Implicit Form** **230**  
 a. Plane curves in implicit form, 230 b. Singular points of curves, 236 c. Implicit representation of surfaces, 238
- 3.3 Systems of Functions, Transformations, and Mappings** **241**  
 a. General remarks, 241 b. Curvilinear coordinates, 246 c. Extension to more than two independent variables, 249 d. Differentiation formulae for the inverse functions,

	252 e. Symbolic product of mappings,	
	257 f. General theorem on the inversion of transformations and of systems of implicit functions. Decomposition into primitive mappings, 261	
	g. Alternate construction of the inverse mapping by the method of successive approximations, 266	
	h. Dependent functions, 268	
	i. Concluding remarks, 275	
<b>3.4</b>	<b>Applications</b>	<b>278</b>
	a. Elements of the theory of surfaces, 278	
	b. Conformal transformation in general, 289	
<b>3.5</b>	<b>Families of Curves, Families of Surfaces, and Their Envelopes</b>	<b>290</b>
	a. General remarks, 290	
	b. Envelopes of one-parameter families of curves, 292	
	c. Examples, 296	
	d. Endvelopes of families of surfaces, 303	
<b>3.6</b>	<b>Alternating Differential Forms</b>	<b>307</b>
	a. Definition of alternating differential forms, 307	
	b. Sums and products of differential forms, 310	
	c. Exterior derivatives of differential forms, 312	
	d. Exterior differential forms in arbitrary coordinates, 316	
<b>3.7</b>	<b>Maxima and Minima</b>	<b>325</b>
	a. Necessary conditions, 325	
	b. Examples, 327	
	c. Maxima and minima with subsidiary conditions, 330	
	d. Proof of the method of undetermined multipliers in the simplest case, 334	
	e. Generalization of the method of undetermined multipliers, 337	
	f. Examples, 340	

**APPENDIX**

<b>A.1 Sufficient Conditions for Extreme Values</b>	<b>345</b>
<b>A.2 Numbers of Critical Points Re- lated to Indices of a Vector Field</b>	<b>352</b>
<b>A.3 Singular Points of Plane Curves</b>	<b>360</b>
<b>A.4 Singular Points of Surfaces</b>	<b>362</b>
<b>A.5 Connection Between Euler's and Lagrange's Representation of the motion of a Fluid</b>	<b>363</b>
<b>A.6 Tangential Representation of a Closed Curve and the Isoperi- metric Inequality</b>	<b>365</b>

*Chapter 4 Multiple Integrals*

<b>4.1 Areas in the Plane</b>	<b>367</b>
a. Definition of the Jordan meas- ure of area, <b>367</b> b. A set that does not have an area, <b>370</b> c. Rules for operations with areas, <b>372</b>	
<b>4.2 Double Integrals</b>	<b>374</b>
a. The double integral as a volume, <b>374</b> b. The general anal- ytic concept of the integral, <b>376</b> c. Examples, <b>379</b> d. Notation. Extensions. Fundamental rules, <b>381</b> e. Integral estimates and the mean value theorem, <b>383</b>	
<b>4.3 Integrals over Regions in three and more Dimensions</b>	<b>385</b>

<b>4.4</b>	<b>Space Differentiation. Mass and Density</b>	<b>386</b>
<b>4.5</b>	<b>Reduction of the Multiple Integral to Repeated Single Integrals</b>	<b>388</b>
	a. Integrals over a rectangle, <b>388</b>	
	b. Change of order of integration. Differentiation under the integral sign, <b>390</b>	
	c. Reduction of double integrals to single integrals for more general regions, <b>392</b>	
	d. Extension of the results to regions in several dimensions, <b>397</b>	
<b>4.6</b>	<b>Transformation of Multiple Integrals</b>	<b>398</b>
	a. Transformation of integrals in the plane, <b>398</b>	
	b. Regions of more than two dimensions, <b>403</b>	
<b>4.7</b>	<b>Improper Multiple Integrals</b>	<b>406</b>
	a. Improper integrals of functions over bounded sets, <b>407</b>	
	b. Proof of the general convergence theorem for improper integrals, <b>411</b>	
	c. Integrals over unbounded regions, <b>414</b>	
<b>4.8</b>	<b>Geometrical Applications</b>	<b>417</b>
	a. Elementary calculation of volumes, <b>417</b>	
	b. General remarks on the calculation of volumes. Solids of revolution. Volumes in spherical coordinates, <b>419</b>	
	c. Area of a curved surface, <b>421</b>	
<b>4.9</b>	<b>Physical Applications</b>	<b>431</b>
	a. Moments and center of mass, <b>431</b>	
	b. Moments of inertia, <b>433</b>	
	c. The compound pendulum, <b>436</b>	
	d. Potential of attracting masses, <b>438</b>	

<b>4.10 Multiple Integrals in Curvilinear Coordinates</b>	<b>445</b>
a. Resolution of multiple integrals, 445	
b. Application to areas swept out by moving curves and volumes swept out by moving surfaces. Guldin's formula. The polar planimeter, 448	
<b>4.11 Volumes and Surface Areas in Any Number of Dimensions</b>	<b>453</b>
a. Surface areas and surface integrals in more than three dimensions, 453	
b. Area and volume of the $n$ -dimensional sphere, 455	
c. Generalizations. Parametric Representations, 459	
<b>4.12 Improper Single Integrals as Functions of a Parameter</b>	<b>462</b>
a. Uniform convergence. Continuous dependence on the parameter, 462	
b. Integration and differentiation of improper integrals with respect to a parameter, 466	
c. Examples, 469	
d. Evaluation of Fresnel's integrals, 473	
<b>4.13 The Fourier Integral</b>	<b>476</b>
a. Introduction, 476	
b. Examples, 479	
c. Proof of Fourier's integral theorem, 481	
d. Rate of convergence in Fourier's integral theorem, 485	
e. Parseval's identity for Fourier transforms, 488	
f. The Fourier transformation for functions of several variables, 490	
<b>4.14 The Eulerian Integrals (Gamma Function)</b>	<b>497</b>
a. Definition and functional equa-	

tion, 497   **b.** Convex functions. Proof of Bohr and Mollerup's theorem, 499   **c.** The infinite products for the gamma function, 503   **d.** The nextensio theorem, 507   **e.** The beta function, 508   **f.** Differentiation and integration of fractional order. Abel's integral equation, 511

**APPENDIX: DETAILED ANALYSIS OF THE PROCESS OF INTEGRATION**

<b>A.1 Area</b>	<b>515</b>
<p><b>a.</b> Subdivisions of the plane and the corresponding inner and outer areas, 515   <b>b.</b> Jordan-measurable sets and their areas, 517   <b>c.</b> Basic properties of areas, 519</p>	
<b>A.2 Integrals of Functions of Several Variables</b>	<b>524</b>
<p><b>a.</b> Definition of the integral of a function <math>f(x, y)</math>, 524   <b>b.</b> Integrability of continuous functions and integrals over sets, 526   <b>c.</b> Basic rules for multiple integrals, 528   <b>d.</b> Reduction of multiple integrals to repeated single integrals, 531</p>	
<b>A.3 Transformation of Areas and Integrals</b>	<b>534</b>
<p><b>a.</b> Mappings of sets, 534   <b>b.</b> Transformation of multiple integrals, 539</p>	
<b>A.4 Note on the Definition of the Area of a Curved Surface</b>	<b>540</b>

## *Chapter 5 Relations Between Surface and Volume Integrals*

<b>5.1</b>	<b>Connection Between Line Integrals and Double Integrals in the Plane (The Integral Theorems of Gauss, Stokes, and Green)</b>	<b>543</b>
<b>5.2</b>	<b>Vector Form of the Divergence Theorem. Stokes's Theorem</b>	<b>551</b>
<b>5.3</b>	<b>Formula for Integration by Parts in Two Dimensions. Green's Theorem</b>	<b>556</b>
<b>5.4</b>	<b>The Divergence Theorem Applied to the Transformation of Double Integrals</b>	<b>558</b>
	<b>a. The case of 1-1 mappings, 558</b>	
	<b>b. Transformation of integrals and degree of mapping, 561</b>	
<b>5.5</b>	<b>Area Differentiation. Transformation of <math>\Delta u</math> to Polar Coordinates</b>	<b>565</b>
<b>5.6</b>	<b>Interpretation of the Formulae of Gauss and Stokes by Two-Dimensional Flows</b>	<b>569</b>
<b>5.7</b>	<b>Orientation of Surfaces</b>	<b>575</b>
	<b>a. Orientation of two-dimensional surfaces in three-space, 575</b>	
	<b>b. Orientation of curves on oriented surfaces, 587</b>	
<b>5.8</b>	<b>Integrals of Differential Forms and of Scalars over Surfaces</b>	<b>589</b>
	<b>a. Double integrals over oriented plane regions, 589</b>	
	<b>b. Surface</b>	

	integrals of second-order differential forms, 592	
	c. Relation between integrals of differential forms over oriented surfaces to integrals of scalars over unoriented surfaces, 594	
<b>5.9</b>	<b>Gauss's and Green's Theorems in Space</b>	<b>597</b>
	a. Gauss's theorem, 597	
	b. Application of Gauss's theorem to fluid flow, 602	
	c. Gauss's theorem applied to space forces and surface forces, 605	
	d. Integration by parts and Green's theorem in three dimensions, 607	
	e. Application of Green's theorem to the transformation of $\Delta U$ to spherical coordinates, 608	
<b>5.10</b>	<b>Stokes's Theorem in Space</b>	<b>611</b>
	a. Statement and proof of the theorem, 611	
	b. Interpretation of Stokes's theorem, 615	
<b>5.11</b>	<b>Integral Identities in Higher Dimensions</b>	<b>622</b>
 <b>APPENDIX: GENERAL THEORY OF SURFACES AND OF SURFACE INTEGRALS</b>		
<b>A.1</b>	<b>Surfaces and Surface Integrals in Three dimensions</b>	<b>624</b>
	a. Elementary surfaces, 624	
	b. Integral of a function over an elementary surface, 627	
	c. Oriented elementary surfaces, 629	
	d. Simple surfaces, 631	
	e. Partitions of unity and integrals over simple surfaces, 634	



<b>A.2 The Divergence Theorem</b>	<b>637</b>
a. Statement of the theorem and its invariance, 637	
b. Proof of the theorem, 639	
<b>A.3 Stokes's Theorem</b>	<b>642</b>
<b>A.4 Surfaces and Surface Integrals in Euclidean Spaces of Higher Dimensions</b>	<b>645</b>
a. Elementary surfaces, 645	
b. Integral of a differential form over an oriented elementary surface, 647	
c. Simple m-dimensional surfaces, 648	
<b>A.5 Integrals over Simple Surfaces, Gauss's Divergence Theorem, and the General Stokes Formula in Higher Dimensions</b>	<b>651</b>

## *Chapter 6 Differential Equations*

<b>6.1 The Differential Equations for the Motion of a Particle in Three Dimensions</b>	<b>654</b>
a. The equations of motion, 654	
b. The principle of conservation of energy, 656	
c. Equilibrium. Stability, 659	
d. Small oscillations about a position of equilibrium, 661	
e. Planetary motion, 665	
f. Boundary value problems. The loaded cable and the loaded beam, 672	
<b>6.2 The General Linear Differential Equation of the First Order</b>	<b>678</b>
a. Separation of variables, 678	
b. The linear first-order equation, 680	

<b>6.3</b>	<b>Linear Differential Equations of Higher Order</b>	<b>683</b>
	a. Principle of superposition. General solutions, <b>683</b>	
	b. Homogeneous differential equations of the second order, <b>688</b>	
	c. The non-homogeneous differential equations. Method of variation of parameters, <b>691</b>	
<b>6.4</b>	<b>General Differential Equations of the First Order</b>	<b>697</b>
	a. Geometrical interpretation, <b>697</b>	
	b. The differential equation of a family of curves. Singular solutions. Orthogonal trajectories, <b>699</b>	
	c. Theorem of the existence and uniqueness of the solution, <b>702</b>	
<b>6.5</b>	<b>Systems of Differential Equations and Differential Equations of Higher Order</b>	<b>709</b>
<b>6.6</b>	<b>Integration by the Method of Undermined Coefficients</b>	<b>711</b>
<b>6.7</b>	<b>The Potential of Attracting Charges and Laplace's Equation</b>	<b>713</b>
	a. Potentials of mass distributions, <b>713</b>	
	b. The differential equation of the potential, <b>718</b>	
	c. Uniform double layers, <b>719</b>	
	d. The mean value theorem, <b>722</b>	
	e. Boundary value problem for the circle. Poisson's integral, <b>724</b>	
<b>6.8</b>	<b>Further Examples of Partial Differential Equations from Mathematical Physics</b>	<b>727</b>
	a. The wave equation in one dimension, <b>727</b>	
	b. The wave equation	

in three-dimensional space, 728  
 c. Maxwell's equations in free space, 731

## *Chapter 7 Calculus of Variations*

<b>7.1</b>	<b>Functions and Their Extrema</b>	<b>737</b>
<b>7.2</b>	<b>Necessary conditions for Extreme Values of a Functional</b>	<b>741</b>
	a. Vanishing of the first variation, 741	
	b. Deduction of Euler's differential equation, 743	
	c. Proofs of the fundamental lemmas, 747	
	d. Solution of Euler's differential equation in special cases. Examples, 748	
	e. Identical vanishing of Euler's expression, 752	
<b>7.3</b>	<b>Generalizations</b>	<b>753</b>
	a. Integrals with more than one argument function, 753	
	b. Examples, 755	
	c. Hamilton's principle. Lagrange's equations, 757	
	d. Integrals involving higher derivatives, 759	
	e. Several independent variables, 760	
<b>7.4</b>	<b>Problems Involving Subsidiary Conditions. Lagrange Multipliers</b>	<b>762</b>
	a. Ordinary subsidiary conditions, 762	
	b. Other types of subsidiary conditions, 765	

## *Chapter 8 Functions of a Complex Variable*

<b>8.1</b>	<b>Complex Functions Represented by Power Series</b>	<b>769</b>
	a. Limits and infinite series with complex terms, 769	
	b. Power	

- series, **772** c. Differentiation and integration of power series, **773**  
 d. Examples of power series, **776**
- 8.2 Foundations of the General Theory of Functions of a Complex Variable** **778**  
 a. The postulate of differentiability, **778** b. The simplest operations of the differential calculus, **782**  
 c. Conformal transformation. Inverse functions, **785**
- 8.3 The Integration of Analytic Functions** **787**  
 a. Definition of the integral, **787**  
 b. Cauchy's theorem, **789** c. Applications. The logarithm, the exponential function, and the general power function, **792**
- 8.4 Cauchy's Formula and Its Applications** **797**  
 a. Cauchy's formula, **797** b. Expansion of analytic functions in power series, **799** c. The theory of functions and potential theory, **802**  
 d. The converse of Cauchy's theorem, **803** e. Zeros, poles, and residues of an analytic function, **803**
- 8.5 Applications to Complex Integration (Contour Integration)** **807**  
 a. Proof of the formula (8.22), **807**  
 b. Proof of the formula (8.22), **808** c. Application of the theorem of residues to the integration of rational functions, **809** d. The theorem of residues and linear differential equations with constant coefficients, **812**

8.6 Many-Valued Functions and Analytic Extension	814
<i>List of Biographical Dates</i>	941*
<i>Index</i>	943**

\* page 543 of this edition

\*\* page 545 of this edition