

Arithmetic Geometry

Arithmetic Geometry

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Preface

This volume is the result of a (mainly) instructional conference on arithmetic geometry, held from July 30 through August 10, 1984 at the University of Connecticut in Storrs. This volume contains expanded versions of almost all the instructional lectures given during the conference. In addition to these expository lectures, this volume contains a translation into English of Faltings' seminal paper which provided the inspiration for the conference. We thank Professor Faltings for his permission to publish the translation and Edward Shipz who did the translation.

We thank all the people who spoke at the Storrs conference, both for helping to make it a successful meeting and enabling us to publish this volume. We would especially like to thank David Rohrlich, who delivered the lectures on height functions (Chapter VI) when the second editor was unavoidably detained. In addition to the editors, Michael Artin and John Tate served on the organizing committee for the conference and much of the success of the conference was due to them—our thanks go to them for their assistance.

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December, 1985

G. Cornell
J. H. Silverman

Contents

Contributors	xiii
Introduction	xv
CHAPTER I	
Some Historical Notes	1
GERD FALTINGS	
§1. The Theorems of Mordell and Mordell–Weil	2
§2. Siegel’s Theorem About Integral Points	2
§3. The Proof of the Mordell Conjecture for Function Fields, by Manin and Grauert	4
§4. The New Ideas of Parshin and Arakelov, Relating the Conjectures of Mordell and Shafarevich	4
§5. The Work of Szpiro, Extending This to Positive Characteristic	6
§6. The Theorem of Tate About Endomorphisms of Abelian Varieties over Finite Fields	6
§7. The Work of Zarhin	6
Bibliographic Remarks	7
CHAPTER II	
Finiteness Theorems for Abelian Varieties over Number Fields	9
GERD FALTINGS	
§1. Introduction	9
§2. Semiabelian Varieties	10
§3. Heights	14
§4. Isogenies	17
§5. Endomorphisms	20
§6. Finiteness Theorems	22
References	26
Erratum	27

CHAPTER III	
Group Schemes, Formal Groups, and p -Divisible Groups	29
STEPHEN S. SHATZ	
§1. Introduction	29
§2. Group Schemes, Generalities	30
§3. Finite Group Schemes	37
§4. Commutative Finite Group Schemes	45
§5. Formal Groups	56
§6. p -Divisible Groups	60
§7. Applications of Groups of Type (p, p, \dots, p) to p -Divisible Groups	76
References	78
CHAPTER IV	
Abelian Varieties over \mathbb{C}	79
MICHAEL ROSEN	
(Notes by F. O. McGuinness)	
§0. Introduction	79
§1. Complex Tori	79
§2. Isogenies of Complex Tori	81
§3. Abelian Varieties	83
§4. The Néron–Severi Group and the Picard Group	92
§5. Polarizations and Polarized Abelian Manifolds	95
§6. The Space of Principally Polarized Abelian Manifolds	97
References	100
CHAPTER V	
Abelian Varieties	103
J. S. MILNE	
§1. Definitions	104
§2. Rigidity	104
§3. Rational Maps into Abelian Varieties	105
§4. Review of the Cohomology of Schemes	108
§5. The Seesaw Principle	109
§6. The Theorems of the Cube and the Square	110
§7. Abelian Varieties Are Projective	112
§8. Isogenies	114
§9. The Dual Abelian Variety: Definition	117
§10. The Dual Abelian Variety: Construction	119
§11. The Dual Exact Sequence	120
§12. Endomorphisms	121
§13. Polarizations and the Cohomology of Invertible Sheaves	126
§14. A Finiteness Theorem	127
§15. The Étale Cohomology of an Abelian Variety	128
§16. Pairings	131
§17. The Rosati Involution	137
§18. Two More Finiteness Theorems	140

§19. The Zeta Function of an Abelian Variety	143
§20. Abelian Schemes	145
References	150

CHAPTER VI

The Theory of Height Functions 151

JOSEPH H. SILVERMAN

The Classical Theory of Heights	151
§1. Absolute Values	151
§2. Height on Projective Space	151
§3. Heights on Projective Varieties	153
§4. Heights on Abelian Varieties	156
§5. The Mordell–Weil Theorem	158
Heights and Metrized Line Bundles	161
§6. Metrized Line Bundles on $\text{Spec}(R)$	161
§7. Metrized Line Bundles on Varieties	161
§8. Distance Functions and Logarithmic Singularities	163
References	166

CHAPTER VII

Jacobian Varieties 167

J. S. MILNE

§1. Definitions	167
§2. The Canonical Maps from C to its Jacobian Variety	171
§3. The Symmetric Powers of a Curve	174
§4. The Construction of the Jacobian Variety	179
§5. The Canonical Maps from the Symmetric Powers of C to its Jacobian Variety	182
§6. The Jacobian Variety as Albanese Variety; Autoduality	185
§7. Weil's Construction of the Jacobian Variety	189
§8. Generalizations	192
§9. Obtaining Coverings of a Curve from its Jacobian; Application to Mordell's Conjecture	195
§10. Abelian Varieties Are Quotients of Jacobian Varieties	198
§11. The Zeta Function of a Curve	200
§12. Torelli's Theorem: Statement and Applications	202
§13. Torelli's Theorem: The Proof	204
Bibliographic Notes for Abelian Varieties and Jacobian Varieties	208
References	211

CHAPTER VIII

Néron Models 213

M. ARTIN

§1. Properties of the Néron Model, and Examples	214
§2. Weil's Construction: Proof	221
§3. Existence of the Néron Model: R Strictly Local	223

§4. Projective Embedding	227
§5. Appendix: Prime Divisors	229
References	230
CHAPTER IX	
Siegel Moduli Schemes and Their Compactifications over \mathbb{C}	231
CHING-LI CHAI	
§0. Notations and Conventions	231
§1. The Moduli Functors and Their Coarse Moduli Schemes	232
§2. Transcendental Uniformization of the Moduli Spaces	235
§3. The Satake Compactification	238
§4. Toroidal Compactification	243
§5. Modular Heights	247
References	250
CHAPTER X	
Heights and Elliptic Curves	253
JOSEPH H. SILVERMAN	
§1. The Height of an Elliptic Curve	253
§2. An Estimate for the Height	256
§3. Weil Curves	260
§4. A Relation with the Canonical Height	263
References	264
CHAPTER XI	
Lipman's Proof of Resolution of Singularities for Surfaces	267
M. ARTIN	
§1. Introduction	267
§2. Proper Intersection Numbers and the Vanishing Theorem	270
§3. Step 1: Reduction to Rational Singularities	274
§4. Basic Properties of Rational Singularities	278
§5. Step 2: Blowing Up the Dualizing Sheaf	281
§6. Step 3: Resolution of Rational Double Points	283
References	287
CHAPTER XII	
An Introduction to Arakelov Intersection Theory	289
T. CHINBURG	
§1. Definition of the Arakelov Intersection Pairing	289
§2. Metrized Line Bundles	292
§3. Volume Forms	294
§4. The Riemann–Roch Theorem and the Adjunction Formula	299
§5. The Hodge Index Theorem	304
References	307
CHAPTER XIII	
Minimal Models for Curves over Dedekind Rings	309
T. CHINBURG	
§1. Statement of the Minimal Models Theorem	309
§2. Factorization Theorem	311

§3. Statement of the Castelnuovo Criterion	314
§4. Intersection Theory and Proper and Total Transforms	315
§5. Exceptional Curves	317
5A. Intersection Properties	317
5B. Prime Divisors Satisfying the Castelnuovo Criterion	319
§6. Proof of the Castelnuovo Criterion	321
§7. Proof of the Minimal Models Theorem	323
References	325

CHAPTER XIV

Local Heights on Curves 327

BENEDICT H. GROSS

§1. Definitions and Notations	327
§2. Néron's Local Height Pairing	328
§3. Construction of the Local Height Pairing	329
§4. The Canonical Height	331
§5. Local Heights for Divisors with Common Support	332
§6. Local Heights for Divisors of Arbitrary Degree	333
§7. Local Heights on Curves of Genus Zero	334
§8. Local Heights on Elliptic Curves	335
§9. Green's Functions on the Upper Half-Plane	336
§10. Local Heights on Mumford Curves	337
References	339

CHAPTER XV

A Higher Dimensional Mordell Conjecture 341

PAUL VOJTA

§1. A Brief Introduction to Nevanlinna Theory	341
§2. Correspondence with Number Theory	344
§3. Higher Dimensional Nevanlinna Theory	347
§4. Consequences of the Conjecture	349
§5. Comparison with Faltings' Proof	352
References	353

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Introduction

The chapters of this book, with the exception of Chapters II, XI and XII, are expanded versions of the lectures given at the Storrs Conference. They are intended, as was the conference, to introduce many of the ideas and techniques currently being used in arithmetic geometry; and in particular to explicate the tools used by Faltings in his proof of the Isogeny, Shafarevich and Mordell conjectures.

The first chapter is a brief overview, by Faltings himself, of the history leading up to the proof of the Mordell conjecture, and the second is a translation from the German of Faltings' paper in which he proved all three conjectures. The heart of this book, Chapters III through IX, contain (with varying amounts of detail) all of the results used in Faltings' paper. In particular, there is a thorough treatment of finite group schemes and p -divisible groups (Chapter III), Abelian and Jacobian varieties and schemes (Chapters IV, V, VII and VIII), their moduli spaces (Chapter IX) and height functions (Chapter VI). The prerequisites vary for each chapter, but in general, little is needed beyond what would normally be covered in one-year graduate courses in algebraic number theory and algebraic geometry.

After a brief chapter to illustrate the general theory for the particular case of elliptic curves (Chapter X), there are four chapters devoted to the theory of local height functions and arithmetic (Arakelov) intersection theory. Finally, Chapter XV contains an exposition of Vojta's far-reaching conjecture, whose consequences would include many of the standard finiteness theorems and outstanding conjectures in arithmetic geometry.

The editors hope that this volume will provide a path into the forest that is modern arithmetic geometry, wherein you will discover the beautiful flowers that blossom when arithmetic and geometry are intertwined, and there perchance, discover some new, exotic species heretofore unknown to the world of mathematics.