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The Kepler Problem

Group Theoretical Aspects,
Regularization and Quantization,
with Application to the Study
of Perturbations

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