

*S. Barry Cooper*

---

# **COMPUTABILITY THEORY**

*CRC PRESS*  
*Boca Raton    London    New York    Washington, D.C.*

---

# Contents

<b>Part I</b>	<b>Computability and Unsolvable Problems</b>	<b>1</b>
<b>1</b>	<b>Hilbert and the Origins of Computability Theory</b>	<b>3</b>
1.1	Algorithms and Algorithmic Content . . . . .	3
1.2	Hilbert's Programme . . . . .	5
1.3	Gödel and the Discovery of Incomputability . . . . .	7
1.4	Computability and Unsolvability in the Real World . . . . .	8
<b>2</b>	<b>Models of Computability and the Church–Turing Thesis</b>	<b>11</b>
2.1	The Recursive Functions . . . . .	12
2.2	Church's Thesis and the Computability of Sets and Relations . . . . .	19
2.3	Unlimited Register Machines . . . . .	25
2.4	Turing's Machines . . . . .	34
2.5	Church, Turing, and the Equivalence of Models . . . . .	42
<b>3</b>	<b>Language, Proof and Computable Functions</b>	<b>45</b>
3.1	Peano Arithmetic and Its Models . . . . .	45
3.2	What Functions Can We Describe in a Theory? . . . . .	56
<b>4</b>	<b>Coding, Self-Reference and the Universal Turing Machine</b>	<b>61</b>
4.1	Russell's Paradox . . . . .	61
4.2	Gödel Numberings . . . . .	62
4.3	A Universal Turing Machine . . . . .	65
4.4	The Fixed Point Theorem . . . . .	66
4.5	Computable Approximations . . . . .	67
<b>5</b>	<b>Enumerability and Computability</b>	<b>69</b>
5.1	Basic Notions . . . . .	69
5.2	The Normal Form Theorem . . . . .	73
5.3	Incomputable Sets and the Unsolvability of the Halting Problem for Turing Machines . . . . .	78
5.4	The Busy Beaver Function . . . . .	81
<b>6</b>	<b>The Search for Natural Examples of Incomputable Sets</b>	<b>87</b>
6.1	The Ubiquitous Creative Sets . . . . .	88
6.2	Some Less Natural Examples of Incomputable Sets . . . . .	90

6.3 Hilbert's Tenth Problem and the Search for Really Natural Examples . . . . .	94
<b>7 Comparing Computability and the Ubiquity of Creative Sets</b>	<b>101</b>
7.1 Many–One Reducibility . . . . .	101
7.2 The Non-Computable Universe and Many–One Degrees . . . . .	106
7.3 Creative Sets Revisited . . . . .	112
<b>8 Gödel's Incompleteness Theorem</b>	<b>117</b>
8.1 Semi-Representability and C.E. Sets . . . . .	117
8.2 Incomputability and Gödel's Theorem . . . . .	122
<b>9 Decidable and Undecidable Theories</b>	<b>127</b>
9.1 PA is Undecidable . . . . .	127
9.2 Other Undecidable Theories and Their Many–One Equivalence	128
9.3 Some Decidable Theories . . . . .	132
<b>Part II Incomputability and Information Content</b>	<b>137</b>
<b>10 Computing with Oracles</b>	<b>139</b>
10.1 Oracle Turing Machines . . . . .	139
10.2 Relativising, and Listing the Partial Computable Functionals	142
10.3 Introducing the Turing Universe . . . . .	144
10.4 Enumerating with Oracles, and the Jump Operator . . . . .	147
10.5 The Arithmetical Hierarchy and Post's Theorem . . . . .	154
10.6 The Structure of the Turing Universe . . . . .	161
<b>11 Nondeterminism, Enumerations and Polynomial Bounds</b>	<b>173</b>
11.1 Oracles versus Enumerations of Data . . . . .	173
11.2 Enumeration Reducibility and the Scott Model for Lambda Calculus . . . . .	180
11.3 The Enumeration Degrees and the Natural Embedding of the Turing Degrees . . . . .	191
11.4 The Structure of $\mathcal{D}_e$ and the Arithmetical Hierarchy . . . . .	199
11.5 The Medvedev Lattice . . . . .	202
11.6 Polynomial Bounds and $\mathbf{P} =? \mathbf{NP}$ . . . . .	205
<b>Part III More Advanced Topics</b>	<b>217</b>
<b>12 Post's Problem: Immunity and Priority</b>	<b>219</b>
12.1 Information Content and Structure . . . . .	219
12.2 Immunity Properties . . . . .	226
12.3 Approximation and Priority . . . . .	237

12.4 Sacks' Splitting Theorem and Cone Avoidance . . . . .	246
12.5 Minimal Pairs and Extensions of Embeddings . . . . .	252
12.6 The $\Pi_3$ Theory — Information Content Regained . . . . .	260
12.7 Higher Priority and Maximal Sets . . . . .	266
<b>13 Forcing and Category</b>	<b>273</b>
13.1 Forcing in Computability Theory . . . . .	273
13.2 Baire Space, Category and Measure . . . . .	278
13.3 $n$ -Genericity and Applications . . . . .	287
13.4 Forcing with Trees and Minimal Degrees . . . . .	299
<b>14 Applications of Determinacy</b>	<b>311</b>
14.1 Gale–Stewart Games . . . . .	311
14.2 An Upper Cone of Minimal Covers . . . . .	315
14.3 Borel and Projective Determinacy, and the Global Theory of $\mathcal{D}$ . . . . .	318
<b>15 The Computability of Theories</b>	<b>321</b>
15.1 Feferman's Theorem . . . . .	321
15.2 Truth versus Provability . . . . .	323
15.3 Complete Extensions of Peano Arithmetic and $\Pi_1^0$ -Classes . . . . .	324
15.4 The Low Basis Theorem . . . . .	329
15.5 Arslanov's Completeness Criterion . . . . .	331
15.6 A Priority-Free Solution to Post's Problem . . . . .	333
15.7 Randomness . . . . .	335
<b>16 Computability and Structure</b>	<b>343</b>
16.1 Computable Models . . . . .	343
16.2 Computability and Mathematical Structures . . . . .	348
16.3 Effective Ramsey Theory . . . . .	362
16.4 Computability in Analysis . . . . .	371
16.5 Computability and Incomputability in Science . . . . .	379
<b>Further Reading</b>	<b>383</b>
<b>Index</b>	<b>389</b>