
WILLIAM W. HAGER
University of Florida
Gainesville, Florida

Computational Methods in Optimal Control Theory and Practice



SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS
PHILADELPHIA

Copyright © 2025 by the Society for Industrial and Applied Mathematics

10 9 8 7 6 5 4 3 2 1

All rights reserved. Printed in the United States of America. No part of this book may be reproduced, stored, or transmitted in any manner without the written permission of the publisher. For information, write to the Society for Industrial and Applied Mathematics, 3600 Market Street, 6th Floor, Philadelphia, PA 19104-2688 USA.

No warranties, express or implied, are made by the publisher, authors, and their employers that the programs contained in this volume are free of error. They should not be relied on as the sole basis to solve a problem whose incorrect solution could result in injury to person or property. If the programs are employed in such a manner, it is at the user’s own risk and the publisher, authors, and their employers disclaim all liability for such misuse.

Trademarked names may be used in this book without the inclusion of a trademark symbol. These names are used in an editorial context only; no infringement of trademark is intended.

<i>Publications Director</i>	Kivmars H. Bowling
<i>Executive Editor</i>	Elizabeth Greenspan
<i>Acquisitions Editor</i>	Paula Callaghan
<i>Developmental Editor</i>	Rose Kolassiba
<i>Managing Editor</i>	Kelly Thomas
<i>Production Editor</i>	Louis R. Primus
<i>Copy Editor</i>	Susan Fleshman
<i>Production Manager</i>	Rachel Ginder
<i>Production Coordinator</i>	Cally A. Shrader
<i>Compositor</i>	Cheryl Hufnagle
<i>Graphic Designer</i>	Doug Smock

Library of Congress Control Number 2024046264

Contents

List of Figures	ix
List of Tables	xi
Preface	xiii
1 Introduction	1
1.1 Some History	1
1.2 Notation	6
1.3 Local versus Global Optima	8
1.4 Approaches and Discretizations	11
1.5 A Standard Problem Formulation	17
2 Convergence Theory	23
2.1 The Abstract Setting	23
2.2 Contraction Mapping Theory	24
2.3 Picard Existence Theory	29
2.4 An Existence Result for Ordinary Equations	30
2.5 An Existence Result for Generalized Equations	32
3 Truncation Errors in Runge–Kutta Methods	35
3.1 First-Order Optimality Conditions	35
3.2 Optimality Conditions for Runge–Kutta Discretizations	38
3.3 Truncation Error for Euler’s Method	40
3.4 Truncation Error for Euler’s Improved Method	43
3.5 Truncation Error for Symplectic Runge–Kutta Schemes	50
3.6 High-Order Symplectic Runge–Kutta Schemes	56
3.7 Condensed Controls with Second-Order Accuracy	60
4 Convergence Theory for Runge–Kutta Methods	65
4.1 Convergence of Euler’s Method	65
4.2 Convergence of Euler’s Improved Method	76
4.3 Convergence of Symplectic Runge–Kutta Schemes	79
4.4 Convergence of Schemes with Condensed Controls	91
5 Orthogonal Collocation	95
5.1 Polynomial Approximations	95
5.2 Orthogonal Collocation	100
5.3 A Runge–Kutta View of hp -Orthogonal Collocation	104

5.4	Truncation Error for Orthogonal Collocation	110
5.5	Convergence of Orthogonal Collocation	118
6	Endpoint Constraints and Discontinuous Controls	123
6.1	Terminal Constraints without Control	123
6.2	Free Terminal Time	129
6.3	Terminal Constraints with Control	133
6.3.1	Analysis of Perturbed Boundary-Value Problem	135
6.3.2	Objective Derivative Relative to a Switch Point	138
6.4	Examples with Discontinuous Optimal Control	141
6.4.1	Harmonic Oscillator Minimum Time Problem	141
6.4.2	Singular Control Problems	143
6.4.3	Generalized Problems	148
A	Minimum Principle	151
	Bibliography	155
	Index	159

List of Figures

1.1	A mass sliding on a frictionless surface while attached to a wall by a spring.	2
1.2	A plot of p_2 for $\theta = 0$	3
1.3	Plot of the switch curve and a minimum time trajectory.	4
1.4	Plot of solution to obstacle problem.	6
1.5	Plot of the optimal altitude (km) of the plane versus time (s).	10
1.6	Plot of the optimal Mach number of the plane versus height (km).	10
1.7	Plot of the optimal α of the plane versus time (s).	10
1.8	Time mesh in the discretization.	12
1.9	The solution of the model problem (1.16).	14
1.10	$\log_{10} E_N$ versus $\log_{10} h$, where E_N is the maximum discrete state error at the mesh points.	16
2.1	The normal cone when \mathbf{u} lies at the vertex of a cone \mathcal{U} and when \mathcal{U} is a sphere with \mathbf{u} on its boundary.	24
3.1	Plot of the optimal state and control for problem (3.47).	44
5.1	Plot of the zeros of P_{14} , the Legendre polynomial of degree 14.	96
5.2	Plot of the flipped Radau points for $N = 14$, where $\sigma_{14} = 1$ and $\sigma_1 > -1$	103
6.1	Euler approximation to the optimal control for harmonic oscillator.	143
6.2	Trajectory of minimum time harmonic oscillator.	144
6.3	Euler approximation to solution of catalyst mixing problem, $N = 100$	145
6.4	Euler approximation to solution of catalyst mixing problem, $N = 1000$	146
6.5	Euler approximation to catalyst mixing problem for $N = 100$ and different choices of the penalty parameter.	147