

David Eisenbud

Commutative Algebra

with a View Toward Algebraic Geometry

With 90 Illustrations



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