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Jean François Colombeau

Multiplication of Distributions

A tool in mathematics, numerical engineering and theoretical physics

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Introduction

The aim of this book is to present a recent mathematical tool, in a way which is very accessible and free from mathematical techniques. The presentation developed here is in part heuristic, with emphasis on algebraic calculations and numerical recipes that can be easily used for numerical solutions of systems of equations modelling elasticity, elastoplasticity, hydrodynamics, acoustic diffusion, multifluid flows. This mathematical tool has also theoretical consequences such as convergence proofs for numerical schemes, existence - uniqueness theorems for solutions of systems of partial differential equations, unification of various methods for defining multiplications of distributions. These topics are not developed in this book since this would have made it not so elementary. A glimpse on these topics is given in two recent research expository papers : Colombeau [14] in Bull. of A.M.S. and Egorov [1] in Russian Math. Surveys. A detailed and careful self contained exposition on these mathematical applications can be found in Oberguggenberger's recent book [11] " Multiplication of distributions and applications to partial differential equations". A set of references is given concerning both the applied and the theoretical viewpoints. This book is the text of a course in numerical modelling given by the author to graduate students at the Ecole Normale Supérieure de Lyon in the academic years 1989 - 90 and 1990 - 91.

Many basic equations of physics contain, in more or less obvious or hidden ways, products looking like "ambiguous multiplications of distributions" such as products of a discontinuous function f and a Dirac mass centered on a point of discontinuity of f or powers of a Dirac mass. These products do not make sense within classical mathematics (i. e. distribution theory) and usually appear as "ambiguous" when considered from a heuristic or physical viewpoint. The idea developed here is that these statements of equations of physics are basically sound, and that a new mathematical theory of generalized functions is needed to explain and master them. Such a theory was first developed in pure mathematics and then it was used in applications ; the mathematician reader can look at the books Colombeau [2, 3], Part II of Rosinger [1], Biagioni [1] and Oberguggenberger [11].

The ambiguity appearing in equations of physics when these equations involve "heuristic multiplications of distributions" corresponds in our theory to the fact that, when formulated in the weakest way, these equations have an infinite number of possible solutions. This recognition of infinitely many solutions was essentially known and understood without our theory (at least in Quantum Field Theory). To resolve the ambiguities our new setting can suggest more precise formulations of the equations (these more precise formulations do not make sense within distribution theory). On physical ground one chooses one of these more precise formulations in which there is no more ambiguity. This technique is developed in this book on various examples from physics. This gives directly new algebraic formulas and new numerical schemes. When one has algebraic jump formulas (for systems in nonconservation form) then it is an easy further step to transfer this knowledge into numerical schemes of the Godunov type. This last numerical technique - Godunov schemes for systems in nonconservation form (elastoplasticity, multifluid flows) or for nonconservative versions of systems of conservation laws (hydrodynamics) - is the main application developed in this text (chapters 4 and 5).

The book is divided into four parts. Part I (chapters 1 and 2) deals with preliminaries from mathematics and physics. Part II (chapter 3) is a smooth introduction to our theory of generalized functions. Part III (chapters 4, 5, 6) is the main part : there new numerical methods are developed ; for simplicity most of them are presented on one dimensional models, but they extend to the 2 and 3 dimensional problems of industrial use or physical significance ; numerical results are presented and references are given. Part IV is made of various complements.

Now let us describe briefly the contents of each chapter. In chapter 1 we introduce our viewpoint, distribution theory and its limitations, in a way convenient for a reader only aware of the concepts of partial derivatives (of functions of several real variables) and of integrals (of continuous functions). Chapter 2 exposes the main equations of Continuum Mechanics considered in the book (hydrodynamics, elastoplasticity, multifluid flows, linear acoustics). The aim of chapter 3 is to describe this new mathematical tool without giving the precise mathematical definitions : the viewpoint there is that these generalized functions can be manipulated correctly provided one has an intuitive understanding of them and provided one is familiar with their rules of calculation. Chapter 4 deals with the classical (conservative) system of fluid dynamics. No products of distributions appear in it , even in case of shock waves. But, surprisingly, our tool gives new methods for its numerical solution : one transforms it into a simpler, but in nonconservative form, system and then one computes a solution from nonconservative Godunov type schemes. In this case, since the correct solution is known with arbitrary precision it is easy to evaluate the value of the new method (by comparison with the exact solution and with numerical results from classical conservative numerical methods). Chapters 5 and 6 deal with systems containing multiplications of distributions that arise directly from physics : nonlinear systems of elastoplasticity and multifluid flows in chapter 5 and linear systems of acoustics in chapter 6. In chapter 7 we expose in the case of a simple model (a self interacting boson field) the basic heuristic calculations of Quantum Field Theory. This topic has been chosen since Quantum Field Theory is the most famous historic example in which the importance of multiplications of distributions was first recognized. Chapter 8 contains a mathematical introduction to these generalized functions and mathematical definitions.

I am particularly indebted to A. Y. Le Roux and B. Poirée. I was working on the multiplication of distributions from a viewpoint of pure mathematics when we met . Their research work (numerical analysis and engineering, physics) had shown them the need of a multiplication of distributions. They introduced me kindly and smoothly to their problems. This was the origin of the present book. I am also very much indebted to L. Arnaud, F. Berger, H. A. Biagioni, L.S. Chadli, P. De Luca, J. Laurens, A. Noussaïr, M. Oberguggenberger, B. Perrot, I. Zalzali for help in works used in the preparation of this book. The main part of the typing has been done by B. Mauduit to whom I also extend my warmest thanks.

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