

DISCRETE MATHEMATICS AND ITS APPLICATIONS  
Series Editor KENNETH H. ROSEN

**HANDBOOK OF  
ELLIPTIC AND  
HYPERELLIPTIC  
CURVE CRYPTOGRAPHY**

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# Table of Contents

<b>List of Algorithms</b>	<b>xxiii</b>
<b>Preface</b>	<b>xxix</b>
<b>1 Introduction to Public-Key Cryptography</b>	<b>1</b>
1.1 Cryptography	2
1.2 Complexity	2
1.3 Public-key cryptography	5
1.4 Factorization and primality	6
1.4.1 Primality	6
1.4.2 Complexity of factoring	6
1.4.3 RSA	7
1.5 Discrete logarithm systems	8
1.5.1 Generic discrete logarithm systems	8
1.5.2 Discrete logarithm systems with bilinear structure	9
1.6 Protocols	9
1.6.1 Diffie–Hellman key exchange	10
1.6.2 Asymmetric Diffie–Hellman and ElGamal encryption	10
1.6.3 Signature scheme of ElGamal-type	12
1.6.4 Tripartite key exchange	13
1.7 Other problems	14

## *I Mathematical Background*

<b>2 Algebraic Background</b>	<b>19</b>
2.1 Elementary algebraic structures	19
2.1.1 Groups	19
2.1.2 Rings	21
2.1.3 Fields	23
2.1.4 Vector spaces	24
2.2 Introduction to number theory	24
2.2.1 Extension of fields	25
2.2.2 Algebraic closure	27
2.2.3 Galois theory	27
2.2.4 Number fields	29
2.3 Finite fields	31
2.3.1 First properties	31
2.3.2 Algebraic extensions of a finite field	32
2.3.3 Finite field representations	33
2.3.4 Finite field characters	35

<b>3</b>	<b>Background on <math>p</math>-adic Numbers</b>	<b>39</b>
3.1	Definition of $\mathbb{Q}_p$ and first properties	39
3.2	Complete discrete valuation rings and fields	41
3.2.1	First properties	41
3.2.2	Lifting a solution of a polynomial equation	42
3.3	The field $\mathbb{Q}_p$ and its extensions	43
3.3.1	Unramified extensions	43
3.3.2	Totally ramified extensions	43
3.3.3	Multiplicative system of representatives	44
3.3.4	Witt vectors	44
<b>4</b>	<b>Background on Curves and Jacobians</b>	<b>45</b>
4.1	Algebraic varieties	45
4.1.1	Affine and projective varieties	46
4.2	Function fields	51
4.2.1	Morphisms of affine varieties	52
4.2.2	Rational maps of affine varieties	53
4.2.3	Regular functions	54
4.2.4	Generalization to projective varieties	55
4.3	Abelian varieties	55
4.3.1	Algebraic groups	55
4.3.2	Birational group laws	56
4.3.3	Homomorphisms of abelian varieties	57
4.3.4	Isomorphisms and isogenies	58
4.3.5	Points of finite order and Tate modules	60
4.3.6	Background on $\ell$ -adic representations	61
4.3.7	Complex multiplication	63
4.4	Arithmetic of curves	64
4.4.1	Local rings and smoothness	64
4.4.2	Genus and Riemann–Roch theorem	66
4.4.3	Divisor class group	76
4.4.4	The Jacobian variety of curves	77
4.4.5	Jacobian variety of elliptic curves and group law	79
4.4.6	Ideal class group	81
4.4.7	Class groups of hyperelliptic curves	83
<b>5</b>	<b>Varieties over Special Fields</b>	<b>87</b>
5.1	Varieties over the field of complex numbers	87
5.1.1	Analytic varieties	87
5.1.2	Curves over $\mathbb{C}$	89
5.1.3	Complex tori and abelian varieties	92
5.1.4	Isogenies of abelian varieties over $\mathbb{C}$	94
5.1.5	Elliptic curves over $\mathbb{C}$	95
5.1.6	Hyperelliptic curves over $\mathbb{C}$	100
5.2	Varieties over finite fields	108
5.2.1	The Frobenius morphism	109
5.2.2	The characteristic polynomial of the Frobenius endomorphism	109
5.2.3	The theorem of Hasse–Weil for Jacobians	110
5.2.4	Tate’s isogeny theorem	112

<b>6</b>	<b>Background on Pairings</b>	<b>115</b>
6.1	General duality results	115
6.2	The Tate pairing	116
6.3	Pairings over local fields	117
6.3.1	The local Tate pairing	118
6.3.2	The Lichtenbaum pairing on Jacobian varieties	119
6.4	An explicit pairing	122
6.4.1	The Tate–Lichtenbaum pairing	122
6.4.2	Size of the embedding degree	123
<b>7</b>	<b>Background on Weil Descent</b>	<b>125</b>
7.1	Affine Weil descent	125
7.2	The projective Weil descent	127
7.3	Descent by Galois theory	128
7.4	Zariski closed subsets inside of the Weil descent	129
7.4.1	Hyperplane sections	129
7.4.2	Trace zero varieties	130
7.4.3	Covers of curves	131
7.4.4	The GHS approach	131
<b>8</b>	<b>Cohomological Background on Point Counting</b>	<b>133</b>
8.1	General principle	133
8.1.1	Zeta function and the Weil conjectures	134
8.1.2	Cohomology and Lefschetz fixed point formula	135
8.2	Overview of $\ell$ -adic methods	137
8.3	Overview of $p$ -adic methods	138
8.3.1	Serre–Tate canonical lift	138
8.3.2	Monsky–Washnitzer cohomology	139

## *II Elementary Arithmetic*

<b>9</b>	<b>Exponentiation</b>	<b>145</b>
9.1	Generic methods	146
9.1.1	Binary methods	146
9.1.2	Left-to-right $2^k$ -ary algorithm	148
9.1.3	Sliding window method	149
9.1.4	Signed-digit recoding	150
9.1.5	Multi-exponentiation	154
9.2	Fixed exponent	157
9.2.1	Introduction to addition chains	157
9.2.2	Short addition chains search	160
9.2.3	Exponentiation using addition chains	163
9.3	Fixed base point	164
9.3.1	Yao’s method	165
9.3.2	Euclidean method	166
9.3.3	Fixed-base comb method	166

<b>10 Integer Arithmetic</b>	<b>169</b>
10.1 Multiprecision integers.	170
10.1.1 Introduction .	170
10.1.2 Internal representation .	171
10.1.3 Elementary operations.	172
10.2 Addition and subtraction .	172
10.3 Multiplication .	174
10.3.1 Schoolbook multiplication .	174
10.3.2 Karatsuba multiplication .	176
10.3.3 Squaring .	177
10.4 Modular reduction .	178
10.4.1 Barrett method.	178
10.4.2 Montgomery reduction.	180
10.4.3 Special moduli.	182
10.4.4 Reduction modulo several primes	184
10.5 Division .	184
10.5.1 Schoolbook division .	185
10.5.2 Recursive division .	187
10.5.3 Exact division .	189
10.6 Greatest common divisor .	190
10.6.1 Euclid extended gcd .	191
10.6.2 Lehmer extended gcd .	192
10.6.3 Binary extended gcd .	194
10.6.4 Chinese remainder theorem .	196
10.7 Square root .	197
10.7.1 Integer square root .	197
10.7.2 Perfect square detection .	198
<b>11 Finite Field Arithmetic</b>	<b>201</b>
11.1 Prime fields of odd characteristic.	201
11.1.1 Representations and reductions .	202
11.1.2 Multiplication .	202
11.1.3 Inversion and division .	205
11.1.4 Exponentiation.	209
11.1.5 Squares and square roots .	210
11.2 Finite fields of characteristic 2 .	213
11.2.1 Representation .	213
11.2.2 Multiplication .	218
11.2.3 Squaring .	221
11.2.4 Inversion and division .	222
11.2.5 Exponentiation .	225
11.2.6 Square roots and quadratic equations .	228
11.3 Optimal extension fields .	229
11.3.1 Introduction .	229
11.3.2 Multiplication .	231
11.3.3 Exponentiation.	231
11.3.4 Inversion .	233
11.3.5 Squares and square roots .	234
11.3.6 Specific improvements for degrees 3 and 5 .	235

<b>12 Arithmetic of <math>p</math>-adic Numbers</b>	<b>239</b>
12.1 Representation	239
12.1.1 Introduction	239
12.1.2 Computing the Teichmüller modulus	240
12.2 Modular arithmetic	244
12.2.1 Modular multiplication	244
12.2.2 Fast division with remainder	244
12.3 Newton lifting	246
12.3.1 Inverse	247
12.3.2 Inverse square root	248
12.3.3 Square root	249
12.4 Hensel lifting	249
12.5 Frobenius substitution	250
12.5.1 Sparse modulus	251
12.5.2 Teichmüller modulus	252
12.5.3 Gaussian normal basis	252
12.6 Artin–Schreier equations	252
12.6.1 Lercier–Lubicz algorithm	253
12.6.2 Harley’s algorithm	254
12.7 Generalized Newton lifting	256
12.8 Applications	257
12.8.1 Teichmüller lift	257
12.8.2 Logarithm	258
12.8.3 Exponential	259
12.8.4 Trace	260
12.8.5 Norm	261

### ***III Arithmetic of Curves***

<b>13 Arithmetic of Elliptic Curves</b>	<b>267</b>
13.1 Summary of background on elliptic curves	268
13.1.1 First properties and group law	268
13.1.2 Scalar multiplication	271
13.1.3 Rational points	272
13.1.4 Torsion points	273
13.1.5 Isomorphisms	273
13.1.6 Isogenies	277
13.1.7 Endomorphisms	277
13.1.8 Cardinality	278
13.2 Arithmetic of elliptic curves defined over $\mathbb{F}_p$	280
13.2.1 Choice of the coordinates	280
13.2.2 Mixed coordinates	283
13.2.3 Montgomery scalar multiplication	285
13.2.4 Parallel implementations	288
13.2.5 Compression of points	288
13.3 Arithmetic of elliptic curves defined over $\mathbb{F}_{2^d}$	289
13.3.1 Choice of the coordinates	291
13.3.2 Faster doublings in affine coordinates	295

13.3.3	Mixed coordinates . . . . .	296
13.3.4	Montgomery scalar multiplication . . . . .	298
13.3.5	Point halving and applications . . . . .	299
13.3.6	Parallel implementation . . . . .	302
13.3.7	Compression of points. . . . .	302
<b>14</b>	<b>Arithmetic of Hyperelliptic Curves . . . . .</b>	<b>303</b>
14.1	Summary of background on hyperelliptic curves . . . . .	304
14.1.1	Group law for hyperelliptic curves . . . . .	304
14.1.2	Divisor class group and ideal class group . . . . .	306
14.1.3	Isomorphisms and isogenies . . . . .	308
14.1.4	Torsion elements . . . . .	309
14.1.5	Endomorphisms . . . . .	310
14.1.6	Cardinality . . . . .	310
14.2	Compression techniques. . . . .	311
14.2.1	Compression in odd characteristic . . . . .	311
14.2.2	Compression in even characteristic . . . . .	313
14.3	Arithmetic on genus 2 curves over arbitrary characteristic . . . . .	313
14.3.1	Different cases . . . . .	314
14.3.2	Addition and doubling in affine coordinates . . . . .	316
14.4	Arithmetic on genus 2 curves in odd characteristic . . . . .	320
14.4.1	Projective coordinates . . . . .	321
14.4.2	New coordinates in odd characteristic . . . . .	323
14.4.3	Different sets of coordinates in odd characteristic . . . . .	325
14.4.4	Montgomery arithmetic for genus 2 curves in odd characteristic . . . . .	328
14.5	Arithmetic on genus 2 curves in even characteristic . . . . .	334
14.5.1	Classification of genus 2 curves in even characteristic. . . . .	334
14.5.2	Explicit formulas in even characteristic in affine coordinates . . . . .	336
14.5.3	Inversion-free systems for even characteristic when $h_2 \neq 0$ . . . . .	341
14.5.4	Projective coordinates . . . . .	341
14.5.5	Inversion-free systems for even characteristic when $h_2 = 0$ . . . . .	345
14.6	Arithmetic on genus 3 curves . . . . .	348
14.6.1	Addition in most common case . . . . .	348
14.6.2	Doubling in most common case . . . . .	349
14.6.3	Doubling on genus 3 curves for even characteristic when $h(x) = 1$ . . . . .	351
14.7	Other curves and comparison . . . . .	352
<b>15</b>	<b>Arithmetic of Special Curves . . . . .</b>	<b>355</b>
15.1	Koblitz curves . . . . .	355
15.1.1	Elliptic binary Koblitz curves . . . . .	356
15.1.2	Generalized Koblitz curves . . . . .	367
15.1.3	Alternative setup . . . . .	375
15.2	Scalar multiplication using endomorphisms . . . . .	376
15.2.1	GLV method . . . . .	377
15.2.2	Generalizations . . . . .	380
15.2.3	Combination of GLV and Koblitz curve strategies . . . . .	381
15.2.4	Curves with endomorphisms for identity-based parameters. . . . .	382
15.3	Trace zero varieties . . . . .	383
15.3.1	Background on trace zero varieties . . . . .	384
15.3.2	Arithmetic in $G$ . . . . .	385



<b>16 Implementation of Pairings</b>	<b>389</b>
16.1 The basic algorithm.	389
16.1.1 The setting	390
16.1.2 Preparation	391
16.1.3 The pairing computation algorithm	391
16.1.4 The case of nontrivial embedding degree $k$ .	393
16.1.5 Comparison with the Weil pairing	395
16.2 Elliptic curves	396
16.2.1 The basic step.	396
16.2.2 The representation	396
16.2.3 The pairing algorithm	397
16.2.4 Example	397
16.3 Hyperelliptic curves of genus 2	398
16.3.1 The basic step	399
16.3.2 Representation for $k > 2$ .	399
16.4 Improving the pairing algorithm	400
16.4.1 Elimination of divisions	400
16.4.2 Choice of the representation	400
16.4.3 Precomputations	400
16.5 Specific improvements for elliptic curves	400
16.5.1 Systems of coordinates	401
16.5.2 Subfield computations.	401
16.5.3 Even embedding degree	402
16.5.4 Example	403

## ***IV Point Counting***

<b>17 Point Counting on Elliptic and Hyperelliptic Curves</b>	<b>407</b>
17.1 Elementary methods	407
17.1.1 Enumeration	407
17.1.2 Subfield curves	409
17.1.3 Square root algorithms	410
17.1.4 Cartier–Manin operator	411
17.2 Overview of $\ell$ -adic methods	413
17.2.1 Schoof’s algorithm	413
17.2.2 Schoof–Elkies–Atkin’s algorithm	414
17.2.3 Modular polynomials	416
17.2.4 Computing separable isogenies in finite fields of large characteristic	419
17.2.5 Complete SEA algorithm	421
17.3 Overview of $p$ -adic methods	422
17.3.1 Satoh’s algorithm	423
17.3.2 Arithmetic–Geometric–Mean algorithm	434
17.3.3 Kedlaya’s algorithm	449

**18 Complex Multiplication . . . . . 455**

- 18.1 CM for elliptic curves . . . . . 456
  - 18.1.1 Summary of background . . . . . 456
  - 18.1.2 Outline of the algorithm . . . . . 456
  - 18.1.3 Computation of class polynomials . . . . . 457
  - 18.1.4 Computation of norms. . . . . 458
  - 18.1.5 The algorithm . . . . . 459
  - 18.1.6 Experimental results . . . . . 459
- 18.2 CM for curves of genus 2 . . . . . 460
  - 18.2.1 Summary of background . . . . . 462
  - 18.2.2 Outline of the algorithm . . . . . 462
  - 18.2.3 CM-types and period matrices . . . . . 463
  - 18.2.4 Computation of the class polynomials . . . . . 465
  - 18.2.5 Finding a curve . . . . . 467
  - 18.2.6 The algorithm . . . . . 469
- 18.3 CM for larger genera . . . . . 470
  - 18.3.1 Strategy and difficulties in the general case . . . . . 470
  - 18.3.2 Hyperelliptic curves with automorphisms . . . . . 471
  - 18.3.3 The case of genus 3 . . . . . 472

***V Computation of Discrete Logarithms***

**19 Generic Algorithms for Computing Discrete Logarithms . . . . . 477**

- 19.1 Introduction . . . . . 478
- 19.2 Brute force . . . . . 479
- 19.3 Chinese remaindering . . . . . 479
- 19.4 Baby-step giant-step . . . . . 480
  - 19.4.1 Adaptive giant-step width . . . . . 481
  - 19.4.2 Search in intervals and parallelization . . . . . 482
  - 19.4.3 Congruence classes . . . . . 483
- 19.5 Pollard's rho method . . . . . 483
  - 19.5.1 Cycle detection . . . . . 484
  - 19.5.2 Application to DL . . . . . 488
  - 19.5.3 More on random walks. . . . . 489
  - 19.5.4 Parallelization . . . . . 489
  - 19.5.5 Automorphisms of the group . . . . . 490
- 19.6 Pollard's kangaroo method . . . . . 491
  - 19.6.1 The lambda method . . . . . 492
  - 19.6.2 Parallelization . . . . . 493
  - 19.6.3 Automorphisms of the group . . . . . 494

**20 Index Calculus . . . . . 495**

- 20.1 Introduction . . . . . 495
- 20.2 Arithmetical formations . . . . . 496
  - 20.2.1 Examples of formations . . . . . 497
- 20.3 The algorithm . . . . . 498
  - 20.3.1 On the relation search . . . . . 499
  - 20.3.2 Parallelization of the relation search . . . . . 500

20.3.3	On the linear algebra	500
20.3.4	Filtering	503
20.3.5	Automorphisms of the group	505
20.4	An important example: finite fields	506
20.5	Large primes	507
20.5.1	One large prime	507
20.5.2	Two large primes	508
20.5.3	More large primes	509
<b>21</b>	<b>Index Calculus for Hyperelliptic Curves</b>	<b>511</b>
21.1	General algorithm	511
21.1.1	Hyperelliptic involution	512
21.1.2	Adleman–DeMarrais–Huang	512
21.1.3	Enge–Gaudry	516
21.2	Curves of small genus	516
21.2.1	Gaudry’s algorithm	517
21.2.2	Refined factor base	517
21.2.3	Harvesting	518
21.3	Large prime methods	519
21.3.1	Single large prime	520
21.3.2	Double large primes	521
<b>22</b>	<b>Transfer of Discrete Logarithms</b>	<b>529</b>
22.1	Transfer of discrete logarithms to $\mathbb{F}_q$ -vector spaces	529
22.2	Transfer of discrete logarithms by pairings	530
22.3	Transfer of discrete logarithms by Weil descent	530
22.3.1	Summary of background	531
22.3.2	The GHS algorithm	531
22.3.3	Odd characteristic	536
22.3.4	Transfer via covers	538
22.3.5	Index calculus method via hyperplane sections	541

## ***VI Applications***

<b>23</b>	<b>Algebraic Realizations of DL Systems</b>	<b>547</b>
23.1	Candidates for secure DL systems	547
23.1.1	Groups with numeration and the DLP	548
23.1.2	Ideal class groups and divisor class groups	548
23.1.3	Examples: elliptic and hyperelliptic curves	551
23.1.4	Conclusion	553
23.2	Security of systems based on $\text{Pic}_C^0$	554
23.2.1	Security under index calculus attacks	554
23.2.2	Transfers by Galois theory	555
23.3	Efficient systems	557
23.3.1	Choice of the finite field	558
23.3.2	Choice of genus and curve equation	560
23.3.3	Special choices of curves and scalar multiplication	563
23.4	Construction of systems	564

23.4.1	Heuristics of class group orders	564
23.4.2	Finding groups of suitable size	565
23.5	Protocols	569
23.5.1	System parameters	569
23.5.2	Protocols on $\text{Pic}_C^0$	570
23.6	Summary	571
<b>24</b>	<b>Pairing-Based Cryptography</b>	<b>573</b>
24.1	Protocols	573
24.1.1	Multiparty key exchange	574
24.1.2	Identity-based cryptography	576
24.1.3	Short signatures	578
24.2	Realization	579
24.2.1	Supersingular elliptic curves	580
24.2.2	Supersingular hyperelliptic curves	584
24.2.3	Ordinary curves with small embedding degree	586
24.2.4	Performance	589
24.2.5	Hash functions on the Jacobian	590
<b>25</b>	<b>Compositeness and Primality Testing – Factoring</b>	<b>591</b>
25.1	Compositeness tests	592
25.1.1	Trial division	592
25.1.2	Fermat tests	593
25.1.3	Rabin–Miller test	594
25.1.4	Lucas pseudoprime tests	595
25.1.5	BPSW tests	596
25.2	Primality tests	596
25.2.1	Introduction	596
25.2.2	Atkin–Morain ECPP test	597
25.2.3	APRCL Jacobi sum test	599
25.2.4	Theoretical considerations and the AKS test	600
25.3	Factoring	601
25.3.1	Pollard’s rho method	601
25.3.2	Pollard’s $p - 1$ method	603
25.3.3	Factoring with elliptic curves	604
25.3.4	Fermat–Morrison–Brillhart approach	607

## ***VII Realization of Discrete Logarithm Systems***

<b>26</b>	<b>Fast Arithmetic in Hardware</b>	<b>617</b>
26.1	Design of cryptographic coprocessors	618
26.1.1	Design criteria	618
26.2	Complement representations of signed numbers	620
26.3	The operation $XY + Z$	622
26.3.1	Multiplication using left shifts	623
26.3.2	Multiplication using right shifts	624
26.4	Reducing the number of partial products	625
26.4.1	Booth or signed digit encoding	625

26.4.2	Advanced recoding techniques	626
26.5	Accumulation of partial products	627
26.5.1	Full adders	627
26.5.2	Faster carry propagation	628
26.5.3	Analysis of carry propagation	631
26.5.4	Multi-operand operations	633
26.6	Modular reduction in hardware	638
26.7	Finite fields of characteristic 2	641
26.7.1	Polynomial basis	642
26.7.2	Normal basis	643
26.8	Unified multipliers	644
26.9	Modular inversion in hardware	645
<b>27</b>	<b>Smart Cards</b>	<b>647</b>
27.1	History	647
27.2	Smart card properties	648
27.2.1	Physical properties	648
27.2.2	Electrical properties	650
27.2.3	Memory	651
27.2.4	Environment and software	656
27.3	Smart card interfaces	659
27.3.1	Transmission protocols	659
27.3.2	Physical interfaces	663
27.4	Types of smart cards	664
27.4.1	Memory only cards (synchronous cards)	664
27.4.2	Microprocessor cards (asynchronous cards)	665
<b>28</b>	<b>Practical Attacks on Smart Cards</b>	<b>669</b>
28.1	Introduction	669
28.2	Invasive attacks	670
28.2.1	Gaining access to the chip	670
28.2.2	Reconstitution of the layers	670
28.2.3	Reading the memories	671
28.2.4	Probing	671
28.2.5	FIB and test engineers scheme flaws	672
28.3	Non-invasive attacks	673
28.3.1	Timing attacks	673
28.3.2	Power consumption analysis	675
28.3.3	Electromagnetic radiation attacks	682
28.3.4	Differential fault analysis (DFA) and fault injection attacks	683
<b>29</b>	<b>Mathematical Countermeasures against Side-Channel Attacks</b>	<b>687</b>
29.1	Countermeasures against simple SCA	688
29.1.1	Dummy arithmetic instructions	689
29.1.2	Unified addition formulas	694
29.1.3	Montgomery arithmetic	696
29.2	Countermeasures against differential SCA	697
29.2.1	Implementation of DSCA	698
29.2.2	Scalar randomization	699
29.2.3	Randomization of group elements	700

29.2.4	Randomization of the curve equation . . . . .	700
29.3	Countermeasures against Goubin type attacks . . . . .	703
29.4	Countermeasures against higher order differential SCA . . . . .	704
29.5	Countermeasures against timing attacks . . . . .	705
29.6	Countermeasures against fault attacks . . . . .	705
29.6.1	Countermeasures against simple fault analysis . . . . .	706
29.6.2	Countermeasures against differential fault analysis . . . . .	706
29.6.3	Conclusion on fault induction . . . . .	708
29.7	Countermeasures for special curves . . . . .	709
29.7.1	Countermeasures against SSCA on Koblitz curves . . . . .	709
29.7.2	Countermeasures against DSCA on Koblitz curves . . . . .	711
29.7.3	Countermeasures for GLV curves . . . . .	713
<b>30</b>	<b>Random Numbers – Generation and Testing . . . . .</b>	<b>715</b>
30.1	Definition of a random sequence . . . . .	715
30.2	Random number generators . . . . .	717
30.2.1	History . . . . .	717
30.2.2	Properties of random number generators . . . . .	718
30.2.3	Types of random number generators . . . . .	718
30.2.4	Popular random number generators . . . . .	720
30.3	Testing of random number generators . . . . .	722
30.4	Testing a device . . . . .	722
30.5	Statistical (empirical) tests . . . . .	723
30.6	Some examples of statistical models on $\Sigma^n$ . . . . .	725
30.7	Hypothesis testings and random sequences . . . . .	726
30.8	Empirical test examples for binary sequences . . . . .	727
30.8.1	Random walk . . . . .	727
30.8.2	Runs . . . . .	728
30.8.3	Autocorrelation . . . . .	728
30.9	Pseudorandom number generators . . . . .	729
30.9.1	Relevant measures . . . . .	730
30.9.2	Pseudorandom number generators from curves . . . . .	732
30.9.3	Other applications . . . . .	735
<b>References</b>	. . . . .	<b>737</b>