Geometry for the Classroom

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With 512 Illustrations

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Using this Book

This book is designed to give a broad preparation in elementary geometry, as well as closely related topics of a slightly more advanced nature. It is appropriate for use by high school students or as a text for preservice or in-service teachers of elementary geometry. The style of presentation and the modular format are designed to incorporate a flexible methodology for the teaching of geometry, one that can be adapted to different classroom settings. The basic strategy is to develop the few fundamental concepts of elementary geometry, first in intuitive form, and then more rigorously. The rest of the material is then built up out of these concepts through a combination of exposition and "guided discovery" in the problem sections.

The Intuition section of the book has individual sections labeled with "I." The intention here is to develop geometric intuition in order to be able to understand interesting, and perhaps rather complicated, geometric results as quickly as possible. The development is logical, but not axiomatic. We use information from arithmetic, algebra, and everyday spatial experiences without making a particular point of making these into axioms. For example, the notion of "rigid motion," that is, moving a figure around, without stretching, bending, or breaking it, is implicitly used throughout.

The Construction section ("C" pages) introduces recipes for all of the standard ruler-and-compass constructions. It is written in a manner interdependent with the Intuition section, that is, later Construction pages use earlier Intuition pages, and vice versa.

The Proof section ("P" pages) is rigorous and axiomatic, and concentrates on the properties of rigid motion. This reflects the point of view of modern geometry, that a geometry is a set, and a distinguished collection of self-transformations of that set.

Finally, there is a Computer Programs section ("CP" pages). This is included to introduce readers to the various geometric figures, constructions, and rigid motions that can be explored using the language LOGO.

Exercises are included with most sections, and are denoted with an "e." For example, the exercises for section I3 are given in I3e. Exercises include some concepts necessary for later parts of the book. So, for a complete understanding of what follows, all exercises should be done by the reader, with answers recorded in the space provided. For many routine numerical exercises, only one prototypical exercise is given. This exercise is intended as a paradigm -- in the high school classroom setting, the teacher may wish to develop a series of similar excercises as necessary to achieve student mastery. Alternatively, excercises may be taken from other texts or from companion materials developed for use with this book.

A basic year-long high school geometry course could consist in the Intuition and Construction sections, perhaps eliminating some of the more elaborate Intuition pages dealing with non-planar geometry. A more complete course, which comes to grips with the notion of rigorous mathematical proofs, would include the Proof pages. The Computer Program pages should be included as much as technological capabilities and student interest allow.

To use *Geometry for the Classroom*, readers will need a good compass, a good straightedge, and several pieces of tracing paper, as well as the appropriate pages of this book. In the classroom setting, students should enter their answers in pencil so that they can be corrected in class before final inked versions are written. These answers will then be permanently incorporated into the reader's version of the book.

About the Authors

C. Herbert Clemens is a professor of mathematics at the University of Utah. His area of mathematical research is algebraic geometry. He was the recipient of an Alfred Sloan Research Fellowship in 1974, and several National Science Foundation grants for mathematical research. He is the author of several mathematical research papers and a textbook. He was an invited speaker at the International Congress of Mathematicians in 1974, and again in 1986. He was a Parent-Teacher at Rosslyn Heights Elementary School in Salt Lake City from 1979 to 1984, during which he wrote the "Graph Paper Fractions Book." He taught the geometry course at Bryant Intermediate School during the academic year 1985-86. He has served on the Mathematical Sciences Education Board of the National Research Council and on the Committee on Mathematics of the National Research Council. He is also an editor of the Pacific Journal of Mathematics.

Michael A. Clemens attended Bryant Intermediate School from 1984 to 1986. During the academic year 1985-1986 he wrote the first draft of this book from materials presented in the geometry course taught by his father and co-author. He is currently a Ph.D. student in economics at Harvard University.

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