

The Finite Element
Method for
Elliptic Problems






TABLE OF CONTENTS

PREFACE TO THE CLASSICS EDITION.	xv
PREFACE	xix
GENERAL PLAN AND INTERDEPENDENCE TABLE	xxvi
1. ELLIPTIC BOUNDARY VALUE PROBLEMS	1
Introduction	1
1.1. Abstract problems	2
The symmetric case. Variational inequalities	2
The nonsymmetric case. The Lax–Milgram lemma	7
Exercises	9
1.2. Examples of elliptic boundary value problems	10
The Sobolev spaces $H^m(\Omega)$. Green's formulas	10
First examples of second-order boundary value problems	15
The elasticity problem	23
Examples of fourth-order problems: The biharmonic problem, the plate problem	28
Exercises	32
Bibliography and Comments	35
2. INTRODUCTION TO THE FINITE ELEMENT METHOD	36
Introduction	36
2.1. Basic aspects of the finite element method	37
The Galerkin and Ritz methods	37
The three basic aspects of the finite element method. Conforming finite element methods	38
Exercises	43
2.2. Examples of finite elements and finite element spaces	43
Requirements for finite element spaces	43
First examples of finite elements for second order problems: n -Simplices of type (k) , $(3')$	44
Assembly in triangulations. The associated finite element spaces n -Rectangles of type (k) . Rectangles of type $(2')$, $(3')$. Assembly in triangulations	51
First examples of finite elements with derivatives as degrees of freedom: Hermite n -simplices of type (3) , $(3')$. Assembly in triangulations	64
First examples of finite elements for fourth-order problems: the	

Argyris and Bell triangles, the Bogner–Fox–Schmit rectangle. Assembly in triangulations	69
Exercises	77
2.3. General properties of finite elements and finite element spaces	78
Finite elements as triples (K, P, Σ) . Basic definitions. The P -interpolation operator	78
Affine families of finite elements	82
Construction of finite element spaces X_h . Basic definitions. The X_h -interpolation operator	88
Finite elements of class \mathcal{C}^0 and \mathcal{C}^1	95
Taking into account boundary conditions. The spaces X_{0h} and X_{00h}	96
Final comments	99
Exercises	101
2.4. General considerations on convergence	103
Convergent family of discrete problems	103
Céa's lemma. First consequences. Orders of convergence	104
Bibliography and comments	106
3. CONFORMING FINITE ELEMENT METHODS FOR SECOND ORDER PROBLEMS	110
Introduction	110
3.1. Interpolation theory in Sobolev spaces	112
The Sobolev spaces $W^{m,p}(\Omega)$. The quotient space $W^{k+1,p}(\Omega)/P_k(\Omega)$	112
Error estimates for polynomial preserving operators	116
Estimates of the interpolation errors $ v - \Pi_K v _{m,q,K}$ for affine families of finite elements	122
Exercises	126
3.2. Application to second-order problems over polygonal domains	131
Estimate of the error $\ u - u_h\ _{1,\Omega}$	131
Sufficient conditions for $\lim_{h \rightarrow 0} \ u - u_h\ _{1,\Omega} = 0$	134
Estimate of the error $ u - u_h _{0,\Omega}$. The Aubin–Nitsche lemma	136
Concluding remarks. Inverse inequalities	139
Exercises	143
3.3. Uniform convergence	147
A model problem. Weighted semi-norms $ \cdot _{\phi,m,\Omega}$	147
Uniform boundedness of the mapping $u \rightarrow u_h$ with respect to appropriate weighted norms	155
Estimates of the errors $ u - u_h _{0,\infty,\Omega}$ and $ u - u_h _{1,\infty,\Omega}$. Nitsche's method of weighted norms	163
Exercises	167
Bibliography and comments	168
4. OTHER FINITE ELEMENT METHODS FOR SECOND-ORDER PROBLEMS	174
Introduction	174
4.1. The effect of numerical integration	178
Taking into account numerical integration. Description of the resulting discrete problem	178
Abstract error estimate: The first Strang lemma	185

Sufficient conditions for uniform V_h -ellipticity	187
Consistency error estimates. The Bramble–Hilbert lemma	190
Estimate of the error $\ u - u_h\ _{1,\Omega}$	199
Exercises	201
4.2. A nonconforming method	207
Nonconforming methods for second-order problems. Description of the resulting discrete problem	207
Abstract error estimate: The second Strang lemma	209
An example of a nonconforming finite element: Wilson’s brick	211
Consistency error estimate. The bilinear lemma	217
Estimate of the error $(\sum_{K \in \mathcal{T}_h} u - u_h _{1,K}^2)^{1/2}$	220
Exercises	223
4.3. Isoparametric finite elements	224
Isoparametric families of finite elements	224
Examples of isoparametric finite elements	227
Estimates of the interpolation errors $ v - \Pi_K v _{m,q,K}$	230
Exercises	243
4.4. Application to second order problems over curved domains	248
Approximation of a curved boundary with isoparametric finite elements	248
Taking into account isoparametric numerical integration. Description of the resulting discrete problem	252
Abstract error estimate	255
Sufficient conditions for uniform V_h -ellipticity	257
Interpolation error and consistency error estimates	260
Estimate of the error $\ \hat{u} - u_h\ _{1,\Omega_h}$	266
Exercises	270
Bibliography and comments	272
Additional bibliography and comments	276
Problems on unbounded domains	276
The Stokes problem	280
Eigenvalue problems	283
5. APPLICATION OF THE FINITE ELEMENT METHOD TO SOME NONLINEAR PROBLEMS	287
Introduction	287
5.1. The obstacle problem	289
Variational formulation of the obstacle problem	289
An abstract error estimate for variational inequalities	291
Finite element approximation with triangles of type (1). Estimate of the error $\ u - u_h\ _{1,\Omega}$	294
Exercises	297
5.2. The minimal surface problem	301
A formulation of the minimal surface problem	301
Finite element approximation with triangles of type (1). Estimate of the error $\ u - u_h\ _{1,\Omega_h}$	302
Exercises	310
5.3. Nonlinear problems of monotone type	312

A minimization problem over the space $W_0^{1,p}(\Omega)$, $2 \leq p$, and its finite element approximation with n -simplices of type (1)	312
Sufficient condition for $\lim_{h \rightarrow 0} \ u - u_h\ _{1,p,\Omega} = 0$	317
The equivalent problem $Au = f$. Two properties of the operator A	318
Strongly monotone operators. Abstract error estimate	321
Estimate of the error $\ u - u_h\ _{1,p,\Omega}$	324
Exercises	324
Bibliography and comments	325
Additional bibliography and comments	330
Other nonlinear problems	330
The Navier–Stokes problem	331
6. FINITE ELEMENT METHODS FOR THE PLATE PROBLEM	333
Introduction	333
6.1. Conforming methods	334
Conforming methods for fourth-order problems	334
Almost-affine families of finite elements	335
A “polynomial” finite element of class \mathcal{C}^1 : The Argyris triangle	336
A composite finite element of class \mathcal{C}^1 : The Hsieh-Clough-Tocher triangle	340
A singular finite element of class \mathcal{C}^1 : The singular Zienkiewicz triangle	347
Estimate of the error $\ u - u_h\ _{2,\Omega}$	352
Sufficient conditions for $\lim_{h \rightarrow 0} \ u - u_h\ _{2,\Omega} = 0$	354
Conclusions	354
Exercises	356
6.2. Nonconforming methods	362
Nonconforming methods for the plate problem	362
An example of a nonconforming finite element: Adini’s rectangle	364
Consistency error estimate. Estimate of the error $(\sum_{K \in \mathcal{T}_h} u - u_h _{2,K}^2)^{1/2}$	367
Further results	373
Exercises	374
Bibliography and comments	376
7. A MIXED FINITE ELEMENT METHOD	381
Introduction	381
7.1. A mixed finite element method for the biharmonic problem	383
Another variational formulation of the biharmonic problem	383
The corresponding discrete problem. Abstract error estimate	386
Estimate of the error $(u - u_h _{1,\Omega} + \Delta u + \phi_h _{0,\Omega})$	390
Concluding remarks	391
Exercise	392
7.2. Solution of the discrete problem by duality techniques	395
Replacement of the constrained minimization problem by a saddle-point problem	395
Use of Uzawa’s method. Reduction to a sequence of discrete Dirichlet problems for the operator $-\Delta$	399

Convergence of Uzawa's method	402
Concluding remarks	403
Exercises	404
Bibliography and comments	406
Additional bibliography and comments	407
Primal, dual and primal-dual formulations	407
Displacement and equilibrium methods	412
Mixed methods	414
Hybrid methods	417
An attempt of general classification of finite element methods	421
8. FINITE ELEMENT METHODS FOR SHELLS	425
Introduction	425
8.1. The shell problem	426
Geometrical preliminaries. Koiter's model	426
Existence of a solution. Proof for the arch problem	431
Exercises	437
8.2. Conforming methods	439
The discrete problem. Approximation of the geometry. Approximation of the displacement	439
Finite element methods conforming for the displacements	440
Consistency error estimates	443
Abstract error estimate	447
Estimate of the error $(\sum_{\alpha=1}^2 \ u_{\alpha} - u_{\alpha h}\ _{1,\Omega}^2 + \ u_3 - u_{3h}\ _{2,\Omega}^2)^{1/2}$	448
Finite element methods conforming for the geometry	450
Conforming finite element methods for shells	450
8.3. A nonconforming method for the arch problem	451
The circular arch problem	451
A natural finite element approximation	452
Finite element methods conforming for the geometry	453
A finite element method which is not conforming for the geometry. Definition of the discrete problem	453
Consistency error estimates	461
Estimate of the error $(u_1 - u_{1h} _{1,I}^2 + u_2 - u_{2h} _{2,I}^2)^{1/2}$	465
Exercise	466
Bibliography and comments	466
EPILOGUE: Some "real-life" finite element model examples	469
BIBLIOGRAPHY	481
GLOSSARY OF SYMBOLS	512
INDEX	521