

**Mathematical
Surveys
and
Monographs**
Volume 110

The Ricci Flow: An Introduction

**Bennett Chow
Dan Knopf**



American Mathematical Society

Contents

Preface	vii
A guide for the reader	viii
A guide for the hurried reader	x
Acknowledgments	xi
Chapter 1. The Ricci flow of special geometries	1
1. Geometrization of three-manifolds	2
2. Model geometries	4
3. Classifying three-dimensional maximal model geometries	6
4. Analyzing the Ricci flow of homogeneous geometries	8
5. The Ricci flow of a geometry with maximal isotropy $SO(3)$	11
6. The Ricci flow of a geometry with isotropy $SO(2)$	15
7. The Ricci flow of a geometry with trivial isotropy	17
Notes and commentary	19
Chapter 2. Special and limit solutions	21
1. Generalized fixed points	21
2. Eternal solutions	24
3. Ancient solutions	28
4. Immortal solutions	34
5. The neckpinch	38
6. The degenerate neckpinch	62
Notes and commentary	66
Chapter 3. Short time existence	67
1. Variation formulas	67
2. The linearization of the Ricci tensor and its principal symbol	71
3. The Ricci–DeTurck flow and its parabolicity	78
4. The Ricci–DeTurck flow in relation to the harmonic map flow	84
5. The Ricci flow regarded as a heat equation	90
Notes and commentary	92
Chapter 4. Maximum principles	93
1. Weak maximum principles for scalar equations	93
2. Weak maximum principles for tensor equations	97
3. Advanced weak maximum principles for systems	100
4. Strong maximum principles	102

Notes and commentary	103
Chapter 5. The Ricci flow on surfaces	105
1. The effect of a conformal change of metric	106
2. Evolution of the curvature	109
3. How Ricci solitons help us estimate the curvature from above	111
4. Uniqueness of Ricci solitons	116
5. Convergence when $\chi(\mathcal{M}^2) < 0$	120
6. Convergence when $\chi(\mathcal{M}^2) = 0$	123
7. Strategy for the case that $\chi(\mathcal{M}^2 > 0)$	128
8. Surface entropy	133
9. Uniform upper bounds for R and $ \nabla R $	137
10. Differential Harnack estimates of LYH type	143
11. Convergence when $R(\cdot, 0) > 0$	148
12. A lower bound for the injectivity radius	149
13. The case that $R(\cdot, 0)$ changes sign	153
14. Monotonicity of the isoperimetric constant	156
15. An alternative strategy for the case $\chi(\mathcal{M}^2 > 0)$	165
Notes and commentary	171
Chapter 6. Three-manifolds of positive Ricci curvature	173
1. The evolution of curvature under the Ricci flow	174
2. Uhlenbeck's trick	180
3. The structure of the curvature evolution equation	183
4. Reduction to the associated ODE system	187
5. Local pinching estimates	189
6. The gradient estimate for the scalar curvature	194
7. Higher derivative estimates and long-time existence	200
8. Finite-time blowup	209
9. Properties of the normalized Ricci flow	212
10. Exponential convergence	218
Notes and commentary	221
Chapter 7. Derivative estimates	223
1. Global estimates and their consequences	223
2. Proving the global estimates	226
3. The Compactness Theorem	231
Notes and commentary	232
Chapter 8. Singularities and the limits of their dilations	233
1. Classifying maximal solutions	233
2. Singularity models	235
3. Parabolic dilations	237
4. Dilations of finite-time singularities	240
5. Dilations of infinite-time singularities	246
6. Taking limits backwards in time	250

Notes and commentary	251
Chapter 9. Type I singularities	253
1. Intuition	253
2. Positive curvature is preserved	255
3. Positive sectional curvature dominates	256
4. Necklike points in finite-time singularities	262
5. Necklike points in ancient solutions	271
6. Type I ancient solutions on surfaces	274
Notes and commentary	277
Appendix A. The Ricci calculus	279
1. Component representations of tensor fields	279
2. First-order differential operators on tensors	280
3. First-order differential operators on forms	283
4. Second-order differential operators	284
5. Notation for higher derivatives	285
6. Commuting covariant derivatives	286
Notes and commentary	286
Appendix B. Some results in comparison geometry	287
1. Some results in local geometry	287
2. Distinguishing between local geometry and global geometry	295
3. Busemann functions	303
4. Estimating injectivity radius in positive curvature	312
Notes and commentary	315
Bibliography	317
Index	323