

Andrej Cherkaev

# Variational Methods for Structural Optimization

With 94 Figures



Springer

Andrej Cherkaev  
Department of Mathematics  
The University of Utah  
Salt Lake City, UT 84112  
USA  
cherk@math.utah.edu

*Editors*

J.E. Marsden  
Control and Dynamical Systems, 107-81  
California Institute of Technology  
Pasadena, CA 91125  
USA

L. Sirovich  
Division of Applied Mathematics  
Brown University  
Providence, RI 02912  
USA

---

Mathematics Subject Classification (1991): 73-02, 73Kxx, 35B27, 49-xx

---

Library of Congress Cataloging-in-Publication Data

Cherkaev, Andrej, 1950–

Variational methods for structural optimization / Andrej Cherkaev.

p. cm. — (Applied mathematical sciences ; 140)

Includes bibliographical references and index.

ISBN 978-1-4612-7038-6 ISBN 978-1-4612-1188-4 (eBook)

DOI 10.1007/978-1-4612-1188-4

1. Structural optimization. 2. Calculus of variations. I. Title. II. Applied mathematical sciences (Springer-Verlag New York, Inc.) ; v. 140.

QA1 .A647 vol. 140

[TA658.8]

510 s—dc21

[624.1'7713]

99-052755

Printed on acid-free paper.

© 2000 Springer-Verlag Berlin Heidelberg

Originally published by Springer-Verlag New York Berlin Heidelberg in 2000

Softcover reprint of the hardcover 1st edition 2000

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer-Verlag Berlin Heidelberg), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden. The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

Production managed by Lesley Poliner; manufacturing supervised by Erica Bressler.

Camera-ready copy prepared from the author's  $\text{\LaTeX}$  files.

9 8 7 6 5 4 3 2 1

ISBN 978-1-4612-7038-6

SPIN 10659110

# Contents

<b>List of Figures</b>	<b>xi</b>
<b>Preface</b>	<b>xv</b>
<b>I Preliminaries</b>	<b>1</b>
<b>1 Relaxation of One-Dimensional Variational Problems</b>	<b>3</b>
1.1 An Optimal Design by Means of Composites . . . . .	3
1.2 Stability of Minimizers and the Weierstrass Test . . . . .	7
1.2.1 Necessary and Sufficient Conditions . . . . .	7
1.2.2 Variational Methods: Weierstrass Test . . . . .	10
1.3 Relaxation . . . . .	14
1.3.1 Nonconvex Variational Problems . . . . .	14
1.3.2 Convex Envelope . . . . .	16
1.3.3 Minimal Extension and Minimizing Sequences . . . . .	19
1.3.4 Examples: Solutions to Nonconvex Problems . . . . .	24
1.3.5 Null-Lagrangians and Convexity . . . . .	27
1.3.6 Duality . . . . .	29
1.4 Conclusion and Problems . . . . .	32
<b>2 Conducting Composites</b>	<b>35</b>
2.1 Conductivity of Inhomogeneous Media . . . . .	35
2.1.1 Equations for Conductivity . . . . .	35
2.1.2 Continuity Conditions in Inhomogeneous Materials . . . . .	39
2.1.3 Energy, Variational Principles . . . . .	42
2.2 Composites . . . . .	45
2.2.1 Homogenization and Effective Tensor . . . . .	46

2.2.2	Effective Properties of Laminates . . . . .	51
2.2.3	Effective Medium Theory: Coated Circles . . . . .	55
2.3	Conclusion and Problems . . . . .	57
<b>3</b>	<b>Bounds and <math>G</math>-Closures</b>	<b>59</b>
3.1	Effective Tensors: Variational Approach . . . . .	59
3.1.1	Calculation of Effective Tensors . . . . .	59
3.1.2	Wiener Bounds . . . . .	61
3.2	$G$ -Closure Problem . . . . .	63
3.2.1	$G$ -convergence . . . . .	63
3.2.2	$G$ -Closure: Definition and Properties . . . . .	67
3.2.3	Example: The $G$ -Closure of Isotropic Materials . . . . .	73
3.2.4	Weak $G$ -Closure (Range of Attainability) . . . . .	75
3.3	Conclusion and Problems . . . . .	76
<b>II</b>	<b>Optimization of Conducting Composites</b>	<b>79</b>
<b>4</b>	<b>Domains of Extremal Conductivity</b>	<b>81</b>
4.1	Statement of the Problem . . . . .	82
4.2	Relaxation Based on the $G$ -Closure . . . . .	83
4.2.1	Relaxation . . . . .	83
4.2.2	Sufficient Conditions . . . . .	85
4.2.3	A Dual Problem . . . . .	89
4.2.4	Convex Envelope and Compatibility Conditions . . . . .	90
4.3	Weierstrass Test . . . . .	92
4.3.1	Variation in a Strip . . . . .	92
4.3.2	The Minimal Extension . . . . .	99
4.3.3	Summary . . . . .	101
4.4	Dual Problem with Nonsmooth Lagrangian . . . . .	103
4.5	Example: The Annulus of Extremal Conductivity . . . . .	108
4.6	Optimal Multiphase Composites . . . . .	110
4.6.1	An Elastic Bar of Extremal Torsion Stiffness . . . . .	110
4.6.2	Multimaterial Design . . . . .	111
4.7	Problems . . . . .	115
<b>5</b>	<b>Optimal Conducting Structures</b>	<b>117</b>
5.1	Relaxation and $G$ -Convergence . . . . .	117
5.1.1	Weak Continuity and Weak Lower Semicontinuity . . . . .	117
5.1.2	Relaxation of Constrained Problems by $G$ -Closure . . . . .	121
5.2	Solution to an Optimal Design Problem . . . . .	123
5.2.1	Augmented Functional . . . . .	123
5.2.2	The Local Problem . . . . .	126
5.2.3	Solution in the Large Scale . . . . .	129
5.3	Reducing to a Minimum Variational Problem . . . . .	130
5.4	Examples . . . . .	134

5.5	Conclusion and Problems . . . . .	139
<b>III</b>	<b>Quasiconvexity and Relaxation</b>	<b>143</b>
<b>6</b>	<b>Quasiconvexity</b>	<b>145</b>
6.1	Structural Optimization Problems . . . . .	145
6.1.1	Statements of Problems of Optimal Design . . . . .	145
6.1.2	Fields and Differential Constraints . . . . .	148
6.2	Convexity of Lagrangians and Stability of Solutions . . . . .	151
6.2.1	Necessary Conditions: Weierstrass Test . . . . .	151
6.2.2	Attainability of the Convex Envelope . . . . .	155
6.3	Quasiconvexity . . . . .	158
6.3.1	Definition of Quasiconvexity . . . . .	158
6.3.2	Quasiconvex Envelope . . . . .	163
6.3.3	Bounds . . . . .	165
6.4	Piecewise Quadratic Lagrangians . . . . .	167
6.5	Problems . . . . .	170
<b>7</b>	<b>Optimal Structures and Laminates</b>	<b>171</b>
7.1	Laminate Bounds . . . . .	171
7.1.1	The Laminate Bound . . . . .	172
7.1.2	Bounds of High Rank . . . . .	174
7.2	Effective Properties of Simple Laminates . . . . .	176
7.2.1	Laminates from Two Materials . . . . .	177
7.2.2	Laminate from a Family of Materials . . . . .	180
7.3	Laminates of Higher Rank . . . . .	182
7.3.1	Differential Scheme . . . . .	183
7.3.2	Matrix Laminates . . . . .	189
7.3.3	Y-Transform . . . . .	193
7.3.4	Calculation of the Fields Inside the Laminates . . . . .	195
7.4	Properties of Complicated Structures . . . . .	198
7.4.1	Multicoated and Self-Repeating Structures . . . . .	198
7.4.2	Structures of Contrast Properties . . . . .	201
7.5	Optimization in the Class of Matrix Composites . . . . .	206
7.6	Discussion and Problems . . . . .	211
<b>8</b>	<b>Lower Bound: Translation Method</b>	<b>213</b>
8.1	Translation Bound . . . . .	213
8.2	Quadratic Translators . . . . .	220
8.2.1	Compensated Compactness . . . . .	220
8.2.2	Determination of Quadratic Translators . . . . .	224
8.3	Translation Bounds for Two-Well Lagrangians . . . . .	228
8.3.1	Basic Formulas . . . . .	228
8.3.2	Extremal Translations . . . . .	229
8.3.3	Example: Lower Bound for the Sum of Energies . . . . .	232

8.3.4	Translation Bounds and Laminate Structures . . . .	235
8.4	Problems . . . . .	237
<b>9</b>	<b>Necessary Conditions and Minimal Extensions</b>	<b>239</b>
9.1	Variational Methods for Nonquasiconvex Lagrangians . . .	239
9.2	Variations . . . . .	241
9.2.1	Variation of Properties . . . . .	241
9.2.2	Increment . . . . .	242
9.2.3	Minimal Extension . . . . .	246
9.3	Necessary Conditions for Two-Phase Composites . . . . .	248
9.3.1	Regions of Stable Solutions . . . . .	248
9.3.2	Minimal Extension . . . . .	249
9.3.3	Necessary Conditions and Compatibility . . . . .	251
9.3.4	Necessary Conditions and Optimal Structures . . . .	253
9.4	Discussion and Problems . . . . .	257
<b>IV</b>	<b><math>G</math>-Closures</b>	<b>259</b>
<b>10</b>	<b>Obtaining <math>G</math>-Closures</b>	<b>261</b>
10.1	Variational Formulation . . . . .	261
10.1.1	Variational Problem for $G_m$ -Closure . . . . .	262
10.1.2	$G$ -Closures . . . . .	269
10.2	The Bounds from Inside by Laminations . . . . .	270
10.2.1	The $L$ -Closure in Two Dimensions . . . . .	274
<b>11</b>	<b>Examples of <math>G</math>-Closures</b>	<b>279</b>
11.1	The $G_m$ -Closure of Two Conducting Materials . . . . .	279
11.1.1	The Variational Problem . . . . .	279
11.1.2	The $G_m$ -Closure in Two Dimensions . . . . .	280
11.1.3	Three-Dimensional Problem . . . . .	284
11.2	$G$ -Closures . . . . .	289
11.2.1	Two Isotropic Materials . . . . .	289
11.2.2	Polycrystals . . . . .	291
11.2.3	Two-Dimensional Polycrystal . . . . .	292
11.2.4	Three-Dimensional Isotropic Polycrystal . . . . .	293
11.3	Coupled Bounds . . . . .	296
11.3.1	Statement of the Problem . . . . .	296
11.3.2	Translation Bounds of $G_m$ -Closure . . . . .	299
11.3.3	The Use of Coupled Bounds . . . . .	305
11.4	Problems . . . . .	308
<b>12</b>	<b>Multimaterial Composites</b>	<b>309</b>
12.1	Special Features of Multicomponent Composites . . . . .	311
12.1.1	Attainability of the Wiener Bound . . . . .	311
12.1.2	Attainability of the Translation Bounds . . . . .	316

12.1.3	The Compatibility of Incompatible Phases . . . . .	321
12.2	Necessary Conditions . . . . .	325
12.2.1	Single Variations . . . . .	326
12.2.2	Composite Variations . . . . .	328
12.3	Optimal Structures for Three-Component Composites . . .	334
12.3.1	Range of Values of the Lagrange Multiplier . . . . .	334
12.3.2	Examples of Optimal Microstructures . . . . .	338
12.4	Discussion . . . . .	341
<b>13</b>	<b>Supplement: Variational Principles for</b>	
	<b>Dissipative Media</b>	<b>343</b>
13.1	Equations of Complex Conductivity . . . . .	344
13.1.1	The Constitutive Relations . . . . .	344
13.1.2	Real Second-Order Equations . . . . .	347
13.2	Variational Principles . . . . .	348
13.2.1	Minimax Variational Principles . . . . .	349
13.2.2	Minimal Variational Principles . . . . .	350
13.3	Legendre Transform . . . . .	352
13.4	Application to $G$ -Closure . . . . .	353
<b>V</b>	<b>Optimization of Elastic Structures</b>	<b>357</b>
<b>14</b>	<b>Elasticity of Inhomogeneous Media</b>	<b>359</b>
14.1	The Plane Problem . . . . .	359
14.1.1	Basic Equations . . . . .	359
14.1.2	Rotation of Fourth-Rank Tensors . . . . .	363
14.1.3	Classes of Equivalency of Elasticity Tensors . . . . .	371
14.2	Three-Dimensional Elasticity . . . . .	373
14.2.1	Equations . . . . .	373
14.2.2	Inhomogeneous Medium. Continuity Conditions . . .	377
14.2.3	Energy, Variational Principles . . . . .	378
14.3	Elastic Structures . . . . .	379
14.3.1	Elastic Composites . . . . .	379
14.3.2	Effective Properties of Elastic Laminates . . . . .	380
14.3.3	Matrix Laminates, Plane Problem . . . . .	382
14.3.4	Three-Dimensional Matrix Laminates . . . . .	385
14.3.5	Ideal Rigid-Soft Structures . . . . .	387
14.4	Problems . . . . .	391
<b>15</b>	<b>Elastic Composites of Extremal Energy</b>	<b>393</b>
15.1	Composites of Minimal Compliance . . . . .	393
15.1.1	The Problem . . . . .	393
15.1.2	Translation Bounds . . . . .	395
15.1.3	Structures . . . . .	398
15.1.4	The Quasiconvex Envelope . . . . .	402

15.1.5	Three-Dimensional Problem . . . . .	403
15.2	Composites of Minimal Stiffness . . . . .	405
15.2.1	Translation Bounds . . . . .	406
15.2.2	The Attainability of the Convex Envelope . . . . .	407
15.3	Optimal Structures Different from Laminates . . . . .	410
15.3.1	Optimal Structures by Vigdergauz . . . . .	410
15.3.2	Optimal Shapes under Shear Loading . . . . .	413
15.4	Problems . . . . .	417
<b>16</b>	<b>Bounds on Effective Properties</b>	<b>419</b>
16.1	$G_m$ -Closures of Special Sets of Materials . . . . .	419
16.2	Coupled Bounds for Isotropic Moduli . . . . .	422
16.2.1	The Hashin–Shtrikman Bounds . . . . .	423
16.2.2	The Translation Bounds . . . . .	425
16.2.3	Functionals . . . . .	429
16.2.4	Translators . . . . .	431
16.2.5	Modification of the Translation Method . . . . .	433
16.2.6	Appendix: Calculation of the Bounds . . . . .	436
16.3	Isotropic Planar Polycrystals . . . . .	447
16.3.1	Bounds . . . . .	448
16.3.2	Extremal Structures: Differential Scheme . . . . .	450
16.3.3	Extremal Structures: Fixed-Point Scheme . . . . .	454
<b>17</b>	<b>Some Problems of Structural Optimization</b>	<b>459</b>
17.1	Properties of Optimal Layouts . . . . .	459
17.1.1	Necessary Conditions . . . . .	460
17.1.2	Remarks on Instabilities . . . . .	463
17.2	Optimization of the Sum of Elastic Energies . . . . .	464
17.2.1	Minimization of the Sum of Elastic Energies . . . . .	465
17.2.2	Optimal Design of Periodic Structures . . . . .	469
17.3	Arbitrary Goal Functionals . . . . .	472
17.3.1	Statement . . . . .	472
17.3.2	Local Problem . . . . .	473
17.3.3	Asymptotics . . . . .	475
17.4	Optimization under Uncertain Loading . . . . .	477
17.4.1	The Formulation . . . . .	477
17.4.2	Eigenvalue Problem . . . . .	480
17.4.3	Multiple Eigenvalues . . . . .	484
17.5	Conclusion . . . . .	491
	<b>References</b>	<b>495</b>
	<b>Author/Editor Index</b>	<b>525</b>
	<b>Subject Index</b>	<b>533</b>