Asymptotic Attainability

Mathematics and Its Applications

Volume 383

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SPRINGER-SCIENCE+BUSINESS MEDIA, B.V.

Library of Congress Cataloging-in-Publication Data

Chenťsov, A. G.
Asymptotic attainability / by A.G. Chentsov.
p. cm. -- (Mathematics and its applications ; v. 383)
Includes index.
ISBN 978-90-481-4765-6 ISBN 978-94-017-0805-0 (eBook)
DOI 10.1007/978-94-017-0805-0
1. Measure theory. 2. Relaxation methods (Mathematics)
3. Extremal problems (Mathematics) I. Title. II. Series:
Mathematics and its applications (Kluwer Academic Publishers) ; v.
383.
QA312.C484 1997
515'.4--dc20 96-36624

ISBN 978-90-481-4765-6

Printed on acid-free paper

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PREFACE

In this monograph, questions of extensions and relaxations are considered. These questions arise in many applied problems in connection with the operation of perturbations. In some cases, the operation of "small" perturbations generates "small" deviations of basis indexes; a corresponding stability takes place. In other cases, small perturbations generate spasmodic change of a result and of solutions defining this result. These cases correspond to unstable problems. The effect of an unstability can arise in extremal problems or in other related problems. In this connection, we note the known problem of constructing the attainability domain in control theory. Of course, extremal problems and those of attainability (in abstract control theory) are connected. We exploit this connection here (see Chapter 5). However, basic attention is paid to the problem of the attainability of elements of a topological space under vanishing perturbations of restrictions. The stability property is frequently missing; the world of unstable problems is of interest for us. We construct regularizing procedures. However, in many cases, it is possible to establish a certain property similar to partial stability. We call this property asymptotic nonsensitivity or roughness under the perturbation of some restrictions. The given property means the following: in the corresponding problem, it is the same if constraints are weakened in some "directions" or not. On this basis, it is possible to construct a certain classification of constraints, selecting "directions of roughness" and "precision directions". Of course, we keep in mind the "directions" in the sense of the operation of perturbations. The realization of elements of this classification is the basic aim of the present monograph. In addition, attention is paid to integral constraints which are used in many statements of extremal problems and, in particular, in problems of control. In these last problems, questions about the investigation of attainability domains play an important role. We consider an abstract analogue of a given concrete problem (recall that the problem of constructing the attainability domain of a controlled system has many engineering applications). Within the framework of the considered general statement, we embrace a series of highly different problems of both "pure" and applied mathematics. For example, we consider the asymptotic behavior of the attainability domain of controlled systems (see Chapters 1 to 4) and questions of the prediction of random events (see Chapter 6). This investigation permits us to establish for the two above-mentioned settings, a series of important analogies to questions connected with the properties of stability and asymptotic nonsensitivity. The common important element of the investigation is the employment of extensions in the class of finitely additive measures (FAM). Convincing reasons exist of this employment.

The corresponding space of FAM with the bounded variation is a Banach space and, what is more, it has a preconjugate Banach space which is that of discontinuous functions. We use the known Alaoglu theorem about the conditions of $*$-weak compactness in a space conjugate to a Banach space. On this basis, highly universal constructions of a compactification are realized. The extension in the class of two-valued $(0,1)$-measures is one such construction. Of course, it is exhaustively characterized in terms of ultrafilters of measure spaces. As a corollary, the given universal construction of an extension considered in Chapter 7 is connected with known procedures of a compactification of topological spaces (in particular, see the Stone-Čech compactification). However, here we concentrate our efforts on constructing some more specialized procedures of an extension. This is connected with many applied problems of control with integral restrictions. For these problems, we design some analogies with generalized functions; in addition, we take into account the possibility of employing discontinuous dependencies in considered conditions. This circumstance requires the application of FAM, which, closing the operation of "usual" controls (integrally bounded functions), have the property of "approximate density" relative to the initial space with a measure. In this connection, the important condition of weak absolute continuity of considered FAM with respect to a given space with a fixed measure arises. In this book, the property of a density of "usual controls" in the space of weakly absolutely continuous FAM is established (see Chapter 3). This statement has the sense of a weakened approximate analogue of the known Radon-Nikodym theorem and embraces cases of an approximation in different topologies. In addition, the weakly absolutely continuous FAM and only they assume the topological approximation by indefinite integrals. However, this statement is distinguished essentially from the known Bochner theorem about the approximation in the strong sense (in the last case, the property of the "usual" absolute continuity is substantial). In the present monograph, "nonstandard" topologies realizing the density properties are introduced. In addition, the construction of a universal "approximate solution" connected with the limiting realization of a weakly absolutely continuous FAM is proposed. Note the very important role of the two following topologies in the space of FAM of bounded variation. The first one is the standard $*$-weak topology; this question is considered below in connection with the compactification of the space of solutions. The second topology corresponds to the Tikhonoff product of samples of a real line in the discrete topology of each such sample. In essence, the following investigation is connected with the application of these topologies in totality. Other topologies of the considered space of FAM are used for auxiliary aims. As a result of the application of different topologies, we obtain an important property of the asymptotic nonsensitivity of the considered problems under the perturbation of an essential part of the conditions. Moreover, under some additional stipulations (having the sense of finiteness of the system of restrictions and of graduatedness of the exploited integrand), we obtain a natural stability.

It is possible to divide this monograph into three parts: 1) Chapters 1–4; 2) Chapter 5; 3) Chapters 6 and 7. In Chapters 1–4, enough simple constructions of the extensions of integral constraints are considered (in Chapters 1 and 2, examples are given). In these chapters, the basic conception of all investigations is explained. In addition, we take into account most typical variants of the perturbations of restrictions. In Chapters 1 and 2, we consider problems of control in the class of ordinary linear differential equations. In the following chapters, we investigate abstract versions of the natural problems of control. This is connected with possible employment of the considered constructions in different applications. We consider approximate solutions (in the general case) as nets in the space of "ordinary" solutions, although in many problems, it is sufficient to exploit only sequential approximate solutions (these cases are considered in part 1); see Chapters 3 and 4). Chapter 5 includes applications of the methods of Chapters 3 and 4 to extremal problems. Finally, part 3) plays the role of a distinctive supplement. In this part, we consider some more general settings; we assume here certain "exotic" perturbations. For example, we exploit some perturbations of the initial measure space. In this part, we consider more compound regularizing constructions in cases when perturbations are not only weakenings of conditions.

It should be noted that the first part (Chapters 1-4) creates an impression about the problem of asymptotic nonsensitivity and the methods of its investigation by the extension within the class of FAM, with a certain property. This property has the sense of a weakened version of the Radon-Nikodym theorem (in the first part, we fix some space with the given FAM on a semi-algebra of sets). The above-mentioned extension is not always reduced to a compactification of the space of solutions. We consider the settings for which admissible sets of the weakened conditions are (generally speaking) unbounded in a strong sense. However, compactifications play an important role. Therefore, we pay sufficient attention to these compactifications. The basis property considered in the present monograph is the asymptotic nonsensitivity under the perturbations of the essential part of restrictions. This property selects rough "directions" in the space of parameters. In addition, there may be no stability in these directions. So, some compensating weakening of conditions is essential for the "correction" of the initial setting of the considered problem. We try to obtain a certain classification of "directions" of the possible intake of perturbations within the system of restrictions. In addition, it is established that many such "directions" are rough in the above-mentioned sense. These properties are established by generalized elements (FAM): we obtain generalized representations (in the class of FAM) universal with respect to different variants of the intake of perturbations. Note that a roughness appears irrespective of the investigated problem. In the case of the problem of attainability in metric spaces, for restrictions including a "resource" part (the constraint in the strong sense) we obtain a neighborhood characterization of asymptotic nonsensitivity. This characterization is most suitable in practical problems. In particular, this circumstance takes place in many problems of control theory. On the other hand, in the general case of an unbounded problem of attainability in a topological space, we characterize the property of asymptotic nonsensitivity in terms of the coincidence of attraction sets. In this case, we have asymptotic nonsensitivity "as a limit"; the latter is natural for asymptotic mathematics. However, in all cases we have a common generalized representation as an instrument for the description of the limit of attainable sets under real perturbations. For this generalized representation, employment of FAM is instrumental.

We analyze questions of an extremum under a perturbation of the conditions of the above-mentioned problems (see Chapter 5). We try to envelop different classes of extremal problems. Attention is focussed on questions of dual constructions for some problems of mathematical programming. These constructions (in principle) permit us to establish approaches to the mathematical realization of the values of the considered problems (see §§5.10, 5.11). Another question connected with numerical realization concerns constructing regularizations. This question is discussed in Chapters 5 and 6. Note that in Chapter 5 problems of multicriteria optimizations and their abstract analogues are considered. Here, a general setting of optimization in a preordered topological space is investigated. In addition, in the given setting, perturbations of restrictions are assumed. An extremum of the corresponding problem is defined as the set of all minimal elements on the space of attainable estimates.

Finally, in the third part of the monograph we investigate the property of asymptotic nonsensitivity in a more general setting. Moreover, we consider here some new aspects of a topological regularization. In addition, new applications of theoretical constructions are discussed. Namely, along with problems of control, we consider (in this part) some problems of processing statistical information. In this connection, problems with stochastic restrictions are considered (see Chapter 6). These settings are connected, in particular, with applied problems for which the processing of samplings is essential. On the basis of these samplings, we try to reconstruct an unknown probability distribution. From this distribution we realize a prediction in the class of mathematical expectations. The above-mentioned prediction is multi-valued, since the initial distribution is reestablished ambiguously.

From processing samplings, we are faced with some errors. In particular, the values of an empirical distribution are different from the true probabilities of corresponding events, even in the case of representative samplings. Moreover, errors arise from the calculation of empirical "average" values . The problems of processing statistical information are (generally speaking) sensitive to these perturbations. In Chapter 6, we investigate constructions of topological regularization of these problems. In addition, we exploit the apparatus of correct extensions in the questions of the "compensation" of a weakening of restrictions, acting by analogy with the above-mentioned problems of control (see Chapters 1–4). We use again FAM with the property of weak absolute continuity for an extension of the space of solutions defined in the form of probability densities. Moreover, we consider regularizations of asymptotic objects by procedures using additional (a priori) information. We analyze constructions of the coordinated employment of extensions and regularizations oriented towards a compensation of nonmonotone perturbations.

In the last Chapter 7, we return to control problems, considering somewhat other settings. Here we note relaxation constructions for problems of impulse control. We again analyze procedures using FAM for the closure of "alternating" mixtures of points from the initial space of solutions. We consider in Chapter 7 some properties of two-valued measures ($(0,1)$-measures) and some other questions connected with the extension of the problem on attainability in nonlinear controlled systems.

The account of theoretical constructions is accompanied by model examples. In Chapter 3, a brief summary of auxiliary notions from topology and functional analysis is given. Moreover, in Chapter 3, definitions and simple properties of FAM are considered.

The bibliography is not complete. Many useful bibliographical references concerned with theory of extension can be found in J. Warga's monograph "Optimal control of functional and differential equations" (1972). In connection with measure theory, we should note the book of J. Distel and J. J. Uhl "Vector measures" (1977) (constructions with employment of vector measures). An extensive consideration of questions of FAM theory is contained in N. Dunford's and J. T. Schwartz's monograph "Linear operators" (1958). We deal with constructions from the author's monograph "Finitely additive measures and relaxations of extremal problems" (in Russian, 1993), using the approach of this monograph from the formalization and investigation of relaxations of extremal problems and problems of attainability under restrictions.

The author expresses his heartfelt thanks to his teacher N. N. Krasovskii for support in the developed direction and for discussing the results. The

works of N. N. Krasovskii exercised the decisive influence on the formation of the author's scientific outlook and were an example of deep penetration into the essence of the problems being solved. They also were an example of overcoming difficulties from the formation of new directions in mathematics. Discussions with A. V. Kryazhimskii and A. I. Subbotin about a wide range of questions connected with extremal problems were very useful to the author. S. I. Morina rendered unvaluable help in translating the manuscript into English and typesetting it, a task that was not easy. The author is grateful for her hard and truly dedicated work. The author also thanks colleagues and scientific workers at the Institute of Mathematics and Mechanics, whose cooperation contributed to a better understanding of problems and the approaches to solving them.

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