# ENUMERATIVE COMBINATORICS

## CHARALAMBOS A. CHARALAMBIDES



A CRC Press Company Boca Raton London New York Washington, D.C.

#### Library of Congress Cataloging-in-Publication Data

Charalambides, Ch. A.
Enumerative combinatorics / Charalambos A. Charalambides.
p. cm. — (The CRC Press series on discrete mathematics and its applications) Includes bibliographical references and index.
ISBN 1-58488-290-5
I. Combinatorial enumeration problems. I. Title. II. Series.

QA164.8 .C48 2002
\$11'.62—dc21
2002019775

This book contains information obtained from authentic and highly regarded sources. Reprinted material is quoted with permission, and sources are indicated. A wide variety of references are listed. Reasonable efforts have been made to publish reliable data and information, but the author and the publisher cannot assume responsibility for the validity of all materials or for the consequences of their use.

Neither this book nor any part may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopying, microfilming, and recording, or by any information storage or retrieval system, without prior permission in writing from the publisher.

The consent of CRC Press LLC does not extend to copying for general distribution, for promotion, for creating new works, or for resale. Specific permission must be obtained in writing from CRC Press LLC for such copying.

Direct all inquiries to CRC Press LLC, 2000 N.W. Corporate Blvd., Boca Raton, Florida 33431.

Trademark Notice: Product or corporate names may be trademarks or registered trademarks, and are used only for identification and explanation, without intent to infringe.

#### Visit the CRC Press Web site at www.crcpress.com

© 2002 by Chapman & Hall/CRC

No claim to original U.S. Government works International Standard Book Number 1-58488-290-5 Library of Congress Card Number 2002019775 Printed in the United States of America 1 2 3 4 5 6 7 8 9 0 Printed on acid-free paper

### **Contents**

Pı	refac	е	v
1	BA	SIC COUNTING PRINCIPLES	1
	1.1	Introduction	1
	1.2	Sets, relations and maps	3
		1.2.1 Basic notions	3
		1.2.2 Cartesian product	5
		1.2.3 Relations	7
		1.2.4 Maps	7
		1.2.5 Countable and uncountable sets	9
		1.2.6 Set operations	9
		1.2.7 Divisions and partitions of a set	13
	1.3	The principles of addition and multiplication	14
	1.4	Discrete probability	24
	1.5	Sums and products	27
	1.6	Bibliographic notes	35
	1.7	Exercises	36
2	PEI	RMUTATIONS AND COMBINATIONS	39
	2.1	Introduction	39
	2.2	Permutations	40
	2.3	Combinations	51
	2.4	Divisions and partitions of a finite set	62
	2.5	Integer solutions of a linear equation	68
	2.6	Lattice paths	75
	2.7	Probabilistic applications	82
		2.7.1 Classical problems in discrete probability	82
		2.7.2 Ordered and unordered samples	86
		2.7.3 Probability models in statistical mechanics	89
	2.8	Bibliographic notes	90
	2.9	Exercises	91

3	FAC	CTORIALS, BINOMIAL AND MULTINOMIAL CO	)_
	EFI	FICIENTS	103
	3.1	Introduction	103
	3.2	Factorials	104
	3.3	Binomial coefficients	110
	3.4	Multinomial coefficients	123
	3.5	Bibliographic notes	124
	3.6	Exercises	124
4	тΗ	E PRINCIPLE OF INCLUSION AND EXCLUSION	131
	4.1	Introduction	131
	4.2	Number of elements in a union of sets	132
	4.3	Number of elements in a given number of sets	144
	4.4	Bonferroni inequalities	152
	4.5	Number of elements of a given rank	155
	4.6	Bibliographic notes	158
	4.7	Exercises	159
5	PE	RMUTATIONS WITH FIXED POINTS AND SUCCE	S-
	SIO	INS	169
	5.1	Introduction	169
	5.2	Permutations with fixed points	169
	5.3	Ranks of permutations	174
	<b>5.4</b>	Permutations with successions	176
	5.5	Circular permutations with successions	180
	5.6	Bibliographic notes	184
	5.7	Exercises	184
6	GE	NERATING FUNCTIONS	191
	6.1	Introduction	191
	6.2	Univariate generating functions	192
		6.2.1 Definitions and basic properties	192
		6.2.2 Power, factorial and Lagrange series	202
	6.3	Combinations and permutations	208
	6.4	Moment generating functions	215
	6.5	Multivariate generating functions	219
	6.6	Bibliographic notes	223
	6.7	Exercises	223
7	RE	CURRENCE RELATIONS	233
	7.1	Introduction	233
	7.2	Basic notions	233
	7.3	Recurrence relations of the first order	235
	7.4	The method of characteristic roots	239

	7.5	The method of generating functions	250
	7.6	Bibliographic notes	264
	7.7	Exercises	264
8	STI	RLING NUMBERS	277
	8.1	Introduction	277
	8.2	Stirling numbers of the first and second kind	278
	8.3	Explicit expressions and recurrence relations	289
	8.4	Generalized factorial coefficients	301
	8.5	Non-central Stirling and related numbers	314
	8.6	Bibliographic notes	319
	8.7	Exercises	321
9	DIS	TRIBUTIONS AND OCCUPANCY	339
	9.1	Introduction	339
	9.2	Classical occupancy and modifications	340
	9.3	Ordered distributions and occupancy	348
	9.4	Balls of general specification and distinguishable urns	350
	9.5	Generating functions	353
	9.6	Bibliographic notes	358
	9.7	Exercises	359
10	PAF	RTITIONS OF INTEGERS	<b>36</b> 9
	10.1	Introduction	369
	10.2	Recurrence relations and generating functions	370
	10.3	A universal generating function $\ldots \ldots \ldots \ldots \ldots \ldots$	376
	10.4	Interrelations among partition numbers	383
	10.5	Combinatorial identities	391
	<b>10.6</b>	Bibliographic notes	396
	10.7	Exercises	396
11	PAF	<b>RTITION POLYNOMIALS</b>	411
	11.1	Introduction	411
	1 <b>1.2</b>	Exponential Bell partition polynomials	412
	11.3	General partition polynomials	419
	11.4	Logarithmic partition polynomials	424
	11.5	Potential partition polynomials	428
	1 <b>1.6</b>	Inversion of power series	433
	11.7	Touchard polynomials	442
	11.8	Bibliographic notes	447
	11.9	Exercises	448

12 CYCLES OF PERMUTATIONS 12.1 Introduction	<b>461</b> 461
12.2 Permutations with a given number of cycles	462
12.3 Even and odd permutations	468
12.4 Permutations with partially ordered cycles	471
12.5 Bibliographic notes	478
12.6 Exercises	478
13 EQUIVALENCE CLASSES	487
13.1 Introduction	487
13.2 Cycle indicator of a permutation group	488
13.3 Orbits of elements of a finite set	493
13.4 Models of colorings of a finite set	499
13.5 Bibliographic notes	507
13.6 Exercises	507
14 RUNS OF PERMUTATIONS AND EULERIAN NUM	-
14 RUNS OF PERMUTATIONS AND EULERIAN NUM BERS	- 513
14 RUNS OF PERMUTATIONS AND EULERIAN NUM BERS         14.1 Introduction	- <b>513</b> 513
14 RUNS OF PERMUTATIONS AND EULERIAN NUM BERS         14.1 Introduction         14.2 Eulerian numbers	- 513 513 513
14 RUNS OF PERMUTATIONS AND EULERIAN NUM BERS         14.1 Introduction         14.2 Eulerian numbers         14.3 Carlitz numbers	- 513 513 513 522
14 RUNS OF PERMUTATIONS AND EULERIAN NUM BERS         14.1 Introduction	<b>513</b> 513 513 522 530
14 RUNS OF PERMUTATIONS AND EULERIAN NUM BERS         14.1 Introduction         14.2 Eulerian numbers         14.3 Carlitz numbers         14.4 Permutations with a given number of runs         14.5 Permutations with repetition and a given number of runs	<b>513</b> 513 513 522 530 533
14 RUNS OF PERMUTATIONS AND EULERIAN NUM BERS         14.1 Introduction         14.2 Eulerian numbers         14.3 Carlitz numbers         14.4 Permutations with a given number of runs         14.5 Permutations with repetition and a given number of runs         14.6 Bibliographic notes	<b>513</b> 513 513 522 530 533 533
14 RUNS OF PERMUTATIONS AND EULERIAN NUM BERS         14.1 Introduction         14.2 Eulerian numbers         14.3 Carlitz numbers         14.4 Permutations with a given number of runs         14.5 Permutations with repetition and a given number of runs         14.6 Bibliographic notes         14.7 Exercises	<b>513</b> 513 513 522 530 533 537 538
14 RUNS OF PERMUTATIONS AND EULERIAN NUM BERS         14.1 Introduction         14.2 Eulerian numbers         14.3 Carlitz numbers         14.4 Permutations with a given number of runs         14.5 Permutations with repetition and a given number of runs         14.6 Bibliographic notes         14.7 Exercises         HINTS AND ANSWERS TO EXERCISES	<b>513</b> 513 513 522 530 533 537 538 <b>545</b>
14 RUNS OF PERMUTATIONS AND EULERIAN NUM BERS         14.1 Introduction         14.2 Eulerian numbers         14.3 Carlitz numbers         14.4 Permutations with a given number of runs         14.5 Permutations with repetition and a given number of runs         14.6 Bibliographic notes         14.7 Exercises         HINTS AND ANSWERS TO EXERCISES         BIBLIOGRAPHY	<b>513</b> 513 522 530 533 537 538 <b>545</b> <b>591</b>