

Riemannian Geometry in an Orthogonal Frame

From lectures delivered by Élie Cartan
at the Sorbonne in 1926–27

Translated from Russian by

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