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James J. Callahan

The Geometry of Spacetime

An Introduction to Special and General Relativity

With 218 Illustrations

Springer

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To Felicity

Preface

Overview

This book is an introduction to Einstein's theories of special and general relativity. To read it, you need only a first course in linear algebra and multivariable calculus and a familiarity with the physical applications of calculus. Because the general theory is more advanced than the special, most books limit themselves to one or the other. However, I have tried to encompass both by using the geometry of spacetime as the unifying theme. Of course, we still have large mathematical bridges to cross. Special relativity is just linear algebra, but general relativity is differential geometry—specifically, the curvature of four-dimensional spacetime.

Origins of the special theory

Einstein's theory of special relativity solved a problem that was baffling physicists at the start of the twentieth century. It concerns what happens when different observers measure the speed of light. Suppose, for example, that one observer moves past a second (stationary) observer at one-tenth the speed of light. We would then expect a light beam moving in the same direction to overtake the moving observer at only nine-tenths of the speed it passes a stationary observer. In fact, careful measurements (the Michelson–Morley experiments in the 1880s) contradict this:

Light moves past all observers at the same speed, independent of their own motion.

To account for this, it was proposed that measuring rods contract slightly when they move and clocks slow down—just enough to make the velocity calculations come out right. In 1895, the Dutch physicist H. A. Lorentz even wrote down the equations that describe how lengths and times must be altered for a moving observer. But these hypotheses were *ad hoc* and just as baffling as the phenomenon they were meant to explain.

The principle of relativity

A decade later, Einstein proposed a solution that was both more radical and more satisfactory. It was built on two assumptions. The first was Galileo's *principle of relativity*:

> Two observers moving uniformly relative to one another must formulate the laws of nature in exactly the same way. In particular, no observer can distinguish between absolute rest and absolute motion by appealing to any law of nature; hence, there is no such thing as absolute motion, but only relative motion (of one observer with respect to another).

Relativity had long lain at the heart of mechanics, but Einstein made it a universal principle that applies to all physical phenomena, including electricity, magnetism, and light. His second assumption was more surprising: Rather than try to explain the invariance of the speed of light, he just accepted it as one of those laws of nature that moving observers must agree upon. From this stance, Einstein then *deduced* the transformation equations of Lorentz and the length contraction and time dilation they entailed. In this new theory of relativity (the adjective "special" came only later, when Einstein introduced a more general theory), the time coordinate is on the same relative footing as the spatial coordinates; furthermore, space and time no longer have separate existences but are fused into a single structure—spacetime. Then, in 1907, the mathematician H. Minkowski showed that Einstein's ideas could be interpreted as a new geometry of spacetime.

Linear spacetime geometry

In Chapter 1 we review the physical problems that prompted the special theory and begin to develop the questions about coordinate transformations that lie at the heart of relativity. Because we assume that observers are in uniform relative motion, the Lorentz transformations that relate their coordinate frames are linear. Geometrically, these transformations are just like spatial rotations, except that their invariant sets are hyperbolas instead of circles. Chapter 2 describes how Einstein made Lorentz transformations the core of a comprehensive theory. We take the geometric viewpoint proposed by Minkowski and develop the Minkowski geometry of spacetime as the invariant theory of Lorentz transformations, making constant comparisons with the familiar Euclidean geometry of ordinary space as the invariant theory of rotations.

We complete the study of special relativity in Chapter 3 by analyzing how objects accelerate in response to imposed forces. Motion here is still governed by Newton's laws, which carry over into spacetime in a straightforward way. We look at the geometric manifestation of acceleration as curvature—in this case, curvature of the curves that objects trace out through spacetime. The chapter also introduces the important *principle of covariance*, which says that physical laws must transform the same way as coordinates.

Origins of the general theory

Special relativity is special because it restricts itself to a small class of observers—those undergoing uniform motion with no acceleration. Their coordinate frames are *inertial*; that is, Galileo's law of inertia holds in them without qualification. The law of inertia is Newton's first law of motion; it says that, in the absence of forces, a body at rest will remain at rest and one in motion will continue to move with constant velocity. The now familiar scenes of astronauts and their equipment floating freely in an orbiting spacecraft show that a frame bound to the spacecraft is inertial. Is the frame of an earthbound laboratory inertial? Objects left to themselves certainly do not float freely, but we explain the motions we see by the force of gravity.

Gravity and general relativity

Our lifelong experience to the contrary notwithstanding, we must regard gravity as a rather peculiar force. It follows from Newton's second law that a given force will impart less acceleration to a large mass than to a small one. But the acceleration of gravity is the same for all masses, so the gravitational force must somehow adjust itself to the mass of each object it pulls. This remarkable property makes it possible to create an artificial gravitational field in space. If we subject a spacecraft far from gravitating masses to constant linear acceleration, then objects inside will "fall down" just as they do on earth. And just as on earth, this artificial force adjusts its strength to give all objects the same downward acceleration. In fact, in any sufficiently small region of spacetime there is no way to distinguish between simple linear acceleration and gravitational acceleration caused by a massive body like the earth. This is Einstein's *principle of equivalence*; he made it the basis of a revolutionary new theory of gravity.

For Einstein, an observer in a gravitational field is simply operating in a certain kind of noninertial frame. If a physical theory is to account for gravity, he reasoned, it must allow noninertial frames on the same footing as inertial ones, and physical laws must take the same form for *all* observers. This is the familiar principle of relativity, but now it is being asserted in its most general form. A successful theory of gravity must be built on general relativity. To help us make the transition from special to general relativity, Chapter 4 considers two kinds of noninertial frames—those that rotate uniformly and those that undergo uniform linear acceleration, from the point of view of an inertial frame. We also survey Newton's theory of gravity and establish both the ordinary differential equations that tell us how a particle moves in a gravitational field and the partial differential equation that tells us how gravitating masses determine the field itself.

Gravity and special relativity are incompatible

The critical discovery in Chapter 4 is that we cannot provide a noninertial frame with the elegant and simple Minkowski geometry we find in a linear spacetime; distances are necessarily distorted. The distortions are the same sort we see in a flat map of a portion of the surface of the earth. Maps distort distances because the earth is curved, so a natural way to explain the distortions that appear when a frame contains a gravitational field is that spacetime is curved. This means that we cannot build an adequate theory of gravity out of Newtonian mechanics and special relativity, because the inertial frames of special relativity are flat.

Curvature is the key to Einstein's theory of gravity, and it is the central topic of Chapters 5 and 6. The simplest circumstance where we can see the essential nature of curvature is in the differential geometry of ordinary surfaces in three-dimensional space—the subject of Chapter 5. At each point a surface has a tangent plane, and each tangent plane has a metric—that is, a way to measure lengths and angles—induced by distance-measurement in the ambient space. With calculus techniques we can then use the metric to do geometric calculations in the surface itself. In this setting, it appears that curvature is an extrinsic feature of a surface's geometry, a manifestation of the way the surface bends in its ambient space. But this setting is both physically and psychologically unsatisfactory, because the four-dimensional spacetime in which we live does not appear to be contained in any larger space that we can perceive. Fortunately, the great nineteenth-century mathematician K. F. Gauss proved that curvature is actually an intrinsic feature of the surface, that is, it can be deduced directly from the metric without reference to the embedding. This opens the way for a more abstract theory of *intrinsic* differential geometry in which a surface patch—and, likewise, the spacetime frame of an arbitrary observer—is simply an open set provided with a suitable metric.

Intrinsic differential geometry

Chapter 6 is about the intrinsic geometry of curved spacetime. It begins with a proof of Gauss's theorem and then goes on to develop the ideas about geodesics and tensors that we need to formulate Einstein's general theory. It explores the fundamental question of relativity: If any two observers describe the same region of curved spacetime, how must their charts G and R be related? The answer is that there is a smooth map $M : G \to R$ whose differential $dM_P : TG_P \to TR_{M(P)}$ is a Lorentz map (that is, a metric-preserving linear map of the tangent spaces) at every point P of G. In other words, special relativity is general relativity "in the small." The nonlinear geometry of spacetime extends the Minkowski geometry of Chapters 2 and 3 in the same way that the nonlinear geometry of surfaces extends Euclidean plane geometry.

Geodesics and the field equations

In Chapter 7 we take up general relativity proper. From the principle of general covariance, Einstein argues that the laws of physics should be expressed as tensor equations if they are to transform properly. Now consider a coordinate frame falling freely in a gravitational field; such a frame is inertial, so an object falling with it moves linearly and thus along a geodesic in that frame. Since geodesics are defined by tensor equations, general covariance guarantees that all observers will say that freely falling objects move on geodesics. Thus, the equations of motion in a gravitational field are the geodesic equations; moreover, the metric in any coordinate frame defines the gravitational field in that frame. The rest of the chapter is devoted to the field equations; these are derived, as they are in the Newtonian theory, from an analysis of tidal forces. Because of the connection between the field and the metric, the field equations tell us not only how the gravitational sources determine the field but how they determine the curvature of spacetime. They summarize Einstein's remarkable conclusion: Gravity is geometry.

The evidence for general relativity

In the final chapter we review the three major pieces of evidence Einstein put forward in support of his theory in the 1916 paper in which he introduced general relativity. First, Einstein demonstrated that general relativity reduces to Newtonian mechanics when the gravitational field is weak and when objects move slowly in relation to the speed of light. The second piece of evidence has to do with the assertion that gravity is curvature. If that is so, a massive object must deflect the path of anything passing it—including a beam of light. Einstein predicted that it should be possible to detect the bending of starlight by the sun during an eclipse; his predictions were fully confirmed in 1919. The third piece of evidence is the precession of the perihelion of Mercury. It was known from the 1860s that the observed value is larger than the value predicted by Newtonian theory; Einstein's theory predicted the observed value with *no* discrepancy. We follow Einstein's arguments and deduce the metric—that is, the gravitational field—associated with a spherically symmetric mass distribution. This involves solving the field equations in two settings; one is the famous Schwarzschild solution and the other is Einstein's own weak-field solution.

The road not taken

My fundamental aim has been to explore the way an individual observer views the world and how ıny pair of observers collaborate to gain objective knowledge of the world. In the simplest case, an observer's coordinate patch is homeomorphic to a ball in $\mathbf{R}^4$, and the tensors the observer uses to formulate physical laws are naturally expressed in terms of the coordinates in that patch. This means that it is not appropriate—at least at the introductory level—to start with a coordinate-free treatment of tensors or to assume that spacetime is a manifold with a potentially complex topology. Indeed, it is by analyzing how any pair of observers must reconcile their individual coordinate descriptions of the physical world that we can see the value and the purpose of these more sophisticated geometric ideas. To keep the text accessible to a reasonably large audience I have also avoided variational methods, even though this has meant using only analogy to justify fundamental results like the relativistic field equations.

Sources

The idea for this book originated in a series of three lectures John Milnor gave at the Universiy of Warwick in the spring of 1978. He showed that it is possible to give a unified picture of relativity in geometric terms for a mathematical audience. His approach was more advanced than the one I have taken here—he used variational methods to formulate some of his key concepts and results—but it began with a development of Minkowski geometry in parallel with Euclidean geometry that was elegant and irresistible. Nearly everything in the lectures was accessible to an undergraduate. For example, Milnor argued that when the relativistic tidal equations are expressed in terms of a Fermi coordinate frame, a symmetric 3×3 matrix appears that corresponds exactly to the matrix used to express the Newtonian tidal equations. The case for the relativistic equations is thereby made by analogy, without recourse to variational arguments.

I have used many other sources as well, but I single out four for particular mention. The first is Einstein's own papers; they appear in English translation along with other valuable papers by Lorentz and Minkowski in *The Principle of Relativity* [20]. Einstein's writing is eminently accessible, and anyone who wants a complete picture of relativity should read his 1916 paper [10]. The other three are more focused on special topics; they are the paper by F. K. Manasse and C. W. Misner on Fermi normal coordinates [21], the treatment of parallel transport in *Geometry from a Differentiable Viewpoint* by John McCleary [22], and the weak-field analysis in *Modern Geometry, Part I* by B. A. Dubrovnin et al. [7].

Since the early 1980s I have taught material in this book in an undergraduate course in either geometry or applied mathematics half a dozen times. My students have always covered Chapters 1–3 and 5 and 6 in some detail and parts of Chapters 4 and 7; we have never had the time to do Chapter 7 thoroughly or Chapter 8 at all. While the text makes progressively greater demands on the reader and the material in the later chapters is more difficult, it is no more difficult than a traditional advanced calculus course. There are points, however, where I have taken advantage of the greater emphasis on differential equations, numerical integration, and computer algebra systems found in the contemporary calculus course.

It would not have been possible for me to write this book without a sabbatical leave and also without the supportive climate over many years that enabled me to develop this material into a course; I am grateful to Smith College and to my colleagues for both. And I particularly want to thank Michael Callahan, whose modern perspective and incisive questions and comments about a number of topics sharpened my thinking about relativity.

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