

1260  
*The Dolciani Mathematical Expositions*

---

NUMBER THIRTY-ONE

# A Garden of Integrals

Frank E. Burk  
*California State University at Chico*

*Published and Distributed by  
The Mathematical Association of America*

# Contents

Foreword	ix
<b>1 An Historical Overview</b>	<b>1</b>
1.1 Rearrangements . . . . .	1
1.2 The Lune of Hippocrates . . . . .	2
1.3 Eudoxus and the Method of Exhaustion . . . . .	4
1.4 Archimedes' Method . . . . .	6
1.5 Gottfried Leibniz and Isaac Newton . . . . .	8
1.6 Augustin-Louis Cauchy . . . . .	11
1.7 Bernhard Riemann . . . . .	12
1.8 Thomas Stieltjes . . . . .	14
1.9 Henri Lebesgue . . . . .	15
1.10 The Lebesgue–Stieltjes Integral . . . . .	18
1.11 Ralph Henstock and Jaroslav Kurzweil . . . . .	19
1.12 Norbert Wiener . . . . .	22
1.13 Richard Feynman . . . . .	26
1.14 References . . . . .	27
<b>2 The Cauchy Integral</b>	<b>29</b>
2.1 Exploring Integration . . . . .	29
2.2 Cauchy's Integral . . . . .	32
2.3 Recovering Functions by Integration . . . . .	35
2.4 Recovering Functions by Differentiation . . . . .	37
2.5 A Convergence Theorem . . . . .	38
2.6 Joseph Fourier . . . . .	40
2.7 P. G. Lejeune Dirichlet . . . . .	41
2.8 Patrick Billingsley's Example . . . . .	43
2.9 Summary . . . . .	44
2.10 References . . . . .	44

<b>3</b>	<b>The Riemann Integral</b>	
3.1	Riemann's Integral	45
3.2	Criteria for Riemann Integrability	45
3.3	Cauchy and Darboux Criteria for Riemann Integrability	49
3.4	Weakening Continuity	52
3.5	Monotonic Functions Are Riemann Integrable	56
3.6	Lebesgue's Criteria	58
3.7	Evaluating à la Riemann	60
3.8	Sequences of Riemann Integrable Functions	63
3.9	The Cantor Set (1883)	65
3.10	A Nowhere Dense Set of Positive Measure	67
3.11	Cantor Functions	69
3.12	Volterra's Example	70
3.13	Lengths of Graphs and the Cantor Function	71
3.14	Summary	72
3.15	References	73
<b>4</b>	<b>The Riemann–Stieltjes Integral</b>	
4.1	Generalizing the Riemann Integral	75
4.2	Discontinuities	75
4.3	Existence of Riemann–Stieltjes Integrals	79
4.4	Monotonicity of $\phi$	80
4.5	Euler's Summation Formula	82
4.6	Uniform Convergence and R-S Integration	83
4.7	References	84
<b>5</b>	<b>Lebesgue Measure</b>	
5.1	Lebesgue's Idea	85
5.2	Measurable Sets	85
5.3	Lebesgue Measurable Sets and Carathéodory	89
5.4	Sigma Algebras	91
5.5	Borel Sets	93
5.6	Approximating Measurable Sets	94
5.7	Measurable Functions	97
5.8	More Measureable Functions	100
5.9	What Does Monotonicity Tell Us?	104
5.10	Lebesgue's Differentiation Theorem	107
5.11	References	109

<b>6</b>	<b>The Lebesgue Integral</b>	<b>111</b>
6.1	Introduction . . . . .	111
6.2	Integrability: Riemann Ensures Lebesgue . . . . .	115
6.3	Convergence Theorems . . . . .	120
6.4	Fundamental Theorems for the Lebesgue Integral . . . . .	127
6.5	Spaces . . . . .	136
6.6	$L^2[-\pi, \pi]$ and Fourier Series . . . . .	148
6.7	Lebesgue Measure in the Plane and Fubini's Theorem . . . . .	151
6.8	Summary . . . . .	152
6.9	References . . . . .	153
<b>7</b>	<b>The Lebesgue–Stieltjes Integral</b>	<b>155</b>
7.1	L-S Measures and Monotone Increasing Functions . . . . .	155
7.2	Carathéodory's Measurability Criterion . . . . .	158
7.3	Avoiding Complacency . . . . .	160
7.4	L-S Measures and Nonnegative Lebesgue Integrable Functions . . . . .	161
7.5	L-S Measures and Random Variables . . . . .	164
7.6	The Lebesgue–Stieltjes Integral . . . . .	165
7.7	A Fundamental Theorem for L-S Integrals . . . . .	166
7.8	Reference . . . . .	167
<b>8</b>	<b>The Henstock–Kurzweil Integral</b>	<b>169</b>
8.1	The Generalized Riemann Integral . . . . .	170
8.2	Gauges and $\delta$ -fine Partitions . . . . .	175
8.3	H-K Integrable Functions . . . . .	176
8.4	The Cauchy Criterion for H-K Integrability . . . . .	183
8.5	Henstock's Lemma . . . . .	187
8.6	Convergence Theorems for the H-K Integral . . . . .	189
8.7	Some Properties of the H-K Integral . . . . .	191
8.8	The Second Fundamental Theorem . . . . .	198
8.9	Summary . . . . .	203
8.10	References . . . . .	204
<b>9</b>	<b>The Wiener Integral</b>	<b>205</b>
9.1	Brownian Motion . . . . .	205
9.2	Construction of the Wiener Measure . . . . .	209
9.3	Wiener's Theorem . . . . .	215
9.4	Measurable Functionals . . . . .	220
9.5	The Wiener Integral . . . . .	222

9.6	Functionals Dependent on a Finite Number of $t$ Values . .	227
9.7	Kac's Theorem . . . . .	232
9.8	References . . . . .	234
10	The Feynman Integral	
10.1	Introduction . . . . .	235
10.2	Summing Probability Amplitudes . . . . .	237
10.3	A Simple Example . . . . .	240
10.4	The Fourier Transform . . . . .	244
10.5	The Convolution Product . . . . .	245
10.6	The Schwartz Space . . . . .	246
10.7	Solving Schrödinger Problem A . . . . .	250
10.8	An Abstract Cauchy Problem . . . . .	252
10.9	Solving in the Schwartz Space . . . . .	254
10.10	Solving Schrödinger Problem B . . . . .	263
10.11	References . . . . .	277
	Index	279
	About the Author	281