

Hans Bühlmann

Mathematical Methods in Risk Theory

 Springer

Hans Bühlmann, Dr. sc. math.
Professor of Mathematics, Eidgenössische Technische Hochschule Zürich

Editors-in-Chief (1970):

Prof. Dr. B. Eckmann
Eidgenössische Technische Hochschule Zürich

Prof. Dr. B. L. van der Waerden
Mathematisches Institut der Universität Zürich

Second printing 2005 (Hardcover)

Library of Congress Control Number: 2005932749

ISSN 0072-7830

ISBN-10 3-540-05117-1 Springer-Verlag Berlin Heidelberg New York

ISBN-13 978-3-540-05117-6 Springer-Verlag Berlin Heidelberg New York

ISBN 3-540-61703-5 2nd-4th Printings (Soft Cover) Springer-Verlag Berlin Heidelberg New York

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable for prosecution under the German Copyright Law.

Springer is a part of Springer Science+Business Media

springeronline.com

© Springer-Verlag Berlin Heidelberg 1970, 2005

Printed in Germany

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Cover design: *design & production* GmbH, Heidelberg
Production: LE-TeX Jelonek, Schmidt & Vöckler GbR, Leipzig

Printed on acid-free paper 41/3142YL - 5 4 3 2 1 0

Table of Contents

Part I. The Theoretical Model

Chapter 1: Probability Aspects of Risk	3
1.1. Random variables explained by the example of claim amount.	3
1.1.1. Definition	3
1.1.2. Classification and examples of distribution functions	4
1.1.3. Expected values	12
1.1.4. Characteristics of a probability distribution and auxiliary functions	15
1.1.5. Chebyshev's Inequality	21
1.2. Sequences of random variables explained by the example of claim amount reproductions	22
1.2.1. Multi-dimensional distributions and auxiliary functions	22
1.2.2. Conditional distribution functions and conditional expectation	25
1.2.3. Independence	28
1.2.4. Covariance and correlation	31
1.2.5. The law of large numbers.	32
Chapter 2: The Risk Process	35
2.1. Fundamentals	35
2.1.1. Definitions and intuitive description of risk	35
2.1.2. Stochastic processes with independent increments	37
2.1.3. Markov processes	39
2.2. The claim number process	41
2.2.1. Mathematical description	41
2.2.2. The claim interoccurrence time	47
2.2.3. The homogeneous claim number process—operational time	49
2.2.4. The case of time-independent intensities of claim frequency: contagion models	51
2.3. The accumulated claim process	54
2.3.1. Definition as random sum and basic representation	54
2.3.2. Proof of the basic representation of the accumulated claim distribution	56
2.3.3. The reduced basic representation: time-independent claim amounts	57
2.3.4. The reduced basic representation: time-dependent claim amounts	58
2.3.5. An example.	60
Chapter 3: The Risk in the Collective	63
3.1. Risk-theoretical definitions	63
3.1.1. Risk and collective	63
3.1.2. The structure function	65
3.2. The weighted risk process as description of the risk in the collective	65
3.2.1. Weighted laws of probability	65
3.2.2. The risk pattern in the collective.	67

3.2.3. The number of claims process in the collective	68
3.2.4. The weighted Poisson and negative binomial distributions	69
3.2.5. The accumulated claim process in the collective	73
3.3. Portfolios in the collective	76
3.3.1. Some definitions	76
3.3.2. Stabilizing in time (Theorem of Ove Lundberg)	77
3.3.3. Stabilizing in size	80
Part II. Consequences of the Theoretical Model	
Chapter 4: Premium Calculation	85
4.1. Principles of premium calculation	85
4.1.1. General	85
4.1.2. Some principles of premium calculation	86
4.1.3. Discussion of the principles of premium calculation	86
4.2. The risk premium and the collective premium	87
4.2.1. The risk premium	87
4.2.2. The collective premium	88
4.2.3. Statistics and collective premium	89
4.2.4. The dilemma and the connection between risk and collective premium	90
4.3. The credibility premium	93
4.3.1. The credibility premium as sequential approximation to the risk premium	93
4.3.2. A new interpretation of the variance principle for calculation of premiums	94
4.3.3. Construction of the credibility premium	96
4.3.4. Assumptions for our further investigations	98
4.3.5. Properties of the credibility premium	98
4.3.6. The credibility formulae for the three components of the credibility premium	100
4.3.7. Determining the weights in the credibility formulae	103
4.4. A practical example: risk, collective and credibility premium in automobile liability insurance	106
Chapter 5: Retentions and Reserves	111
5.1. The retention problem	111
5.1.1. General	111
5.1.2. The retention under proportional and non-proportional reinsurance	112
5.2. The relative retention problem	113
5.2.1. Proportional reinsurance	114
5.2.2. Non-proportional reinsurance	116
5.2.3. The risk with given risk parameter and the risk in the collective under non-proportional reinsurance	119
5.2.4. Credibility approximation for the relative retention	121
5.3. The absolute retention problem	124
5.3.1. Exact statement of the problem	124
5.3.2. The random walk of the risk carrier's free reserves generated by the risk mass	126
5.4. Reserves	129

Chapter 6: The Insurance Carrier's Stability Criteria	131
6.1. The stability problem	131
6.1.1. Decision variables	131
6.1.2. Stability problem and stability criteria	132
6.2. The probability of ruin as stability criterion	133
6.2.1. Planning horizon and ruin probability	133
6.2.2. Admissible stability policies	135
6.2.3. Hypotheses about the model variables in calculating the probability of ruin	135
6.2.4. Calculating the probability of ruin in the discrete case with finite planning horizon	137
6.2.5. Calculating the probability of ruin with an infinite planning horizon using the Wiener-Hopf method	141
6.2.6. Calculating the probability of ruin in the continuous case with infinite planning horizon using renewal theory methods.	144
6.3. The absolute retention when the probability of ruin is chosen as the stability criterion	152
6.3.1. Restatement of the problem and assumptions	152
6.3.2. The optimal gradation of retentions	154
6.3.3. The stability condition	155
6.3.4. Determining the absolute retention when the risk parameter is known	156
6.3.5. Determining the absolute retention when the risk parameters are drawn from one or more collectives	159
6.3.6. Practical remark on the probability of ruin as stability criterion	163
6.4. Dividend policy as criterion of stability	164
6.4.1. General description of the criterion	164
6.4.2. Hypotheses about the model variables when the dividend policy is used as stability criterion	165
6.4.3. Dividend policy in the discrete case	165
6.4.4. Results in the discrete case.	166
6.4.5. Barrier strategies in the discrete case	168
6.4.6. Dividend policy in the continuous case	168
6.4.7. The integro-differential equation of the barrier strategy in the continuous case	171
6.4.8. Solving the integro-differential equation for $V(Q, a)$	172
6.4.9. Asymptotic formula for a_0	174
6.4.10. Optimum dividend policy for $Q > a_0$ and other evaluations.	177
6.5. Utility as criterion of stability	178
6.5.1. Evaluating the random walk of free reserves	178
6.5.2. Equivalent evaluations; definition of utility	179
6.5.3. Axioms about utility	182
6.5.4. Existence theorem for an equivalent utility	184
6.5.5. Integral evaluation.	188
6.5.6. The problem of risk exchange	190
6.5.7. The theorem of Borch	191
6.5.8. A consequence of Borch's theorem.	195
6.5.9. Price structures with quadratic utility kernels	197

Appendix: The Generalized Riemann-Stieltjes Integral	201
A.1. Preliminary	201
A.2. Definition of the generalized Riemann-Stieltjes integral in two special cases	201
A.3. Definition in the general case	203
A.4. Integrable functions	203
A.5. Properties of the generalized Riemann-Stieltjes integral	204
Bibliography	206
Index	209