Cambridge University Press 0521823293 - Approximation by Algebraic Numbers Yann Bugeaud Table of Contents More information

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