Graduate Texts in Mathematics 98

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Theodor Bröcker
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Representations of Compact Lie Groups

With 24 Illustrations

Springer

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Auch ging es mir, wie jedem, der reisend oder lebend mit Ernst gehandelt, daß ich in dem Augenblicke des Scheidens erst einigermaßen mich wert fühlte, hereinzutreten. Mich trösteten die mannigfaltigen und unentwickelten Schätze, die ich mir gesammlet.

G.

Preface

This book is based on several courses given by the authors since 1966. It introduces the reader to the representation theory of compact Lie groups.

We have chosen a geometrical and analytical approach since we feel that this is the easiest way to motivate and establish the theory and to indicate relations to other branches of mathematics. Lie algebras, though mentioned occasionally, are not used in an essential way. The material as well as its presentation are classical; one might say that the foundations were known to Hermann Weyl at least 50 years ago.

Prerequisites to the book are standard linear algebra and analysis, including Stokes' theorem for manifolds. The book can be read by German students in their third year, or by first-year graduate students in the United States.

Generally speaking the book should be useful for mathematicians with geometric interests and, we hope, for physicists.

At the end of each section the reader will find a set of exercises. These vary in character: Some ask the reader to verify statements used in the text, some contain additional information, and some present examples and counter-examples. We advise the reader at least to read through the exercises.

The book is organized as follows. There are six chapters, each containing several sections. A reference of the form III, (6.2) refers to Theorem (Definition, etc.) (6.2) in Section 6 of Chapter III. The roman numeral is omitted whenever the reference concerns the chapter where it appears. References to the Bibliography at the end of the book have the usual form, e.g. Weyl [1].

Naturally, we would have liked to write in our mother tongue. But we hope that our English will be acceptable to a larger mathematical community, although any personal manner may have been lost and we do not feel competent judges on matters of English style.

Arunas Liulevicius, Wolfgang Lück, and Klaus Wirthmüller have read the manuscript and suggested many improvements. We thank them for their generous help. We are most grateful to Robert Robson who translated part of the German manuscript and revised the whole English text.

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