Grundlehren der mathematischen Wissenschaften 319

A Series of Comprehensive Studies in Mathematics

Springer-Verlag Berlin Heidelberg GmbH

Martin R. Bridson
André Haefliger

Metric Spaces of Non-Positive Curvature

With 84 Figures

Springer

Martin R. Bridson
University of Oxfod
Mathematical Institute
24-29 St. Giles'
Oxford OX1 3LB
Great Britain
email: bridson@maths.ox.ac.uk

André Haefliger
Université de Genève
Section de Mathématiques
2-4 Rue du Lièvre, CP-240
CH-1211 Genève 24
Switzerland
email: Andre.Haefliger@math.unige.ch

Library of Congress Cataloging-in-Publication Data
Bridson, Martin R., 1964–
Metric spaces of non-positive curvature / Martin R. Bridson, André Haefliger.
p. cm. – (Grundlehren der mathematischen Wissenschaften, ISSN 0072-7830; 319)
Includes bibliographical references and index.
ISBN 978-3-642-08399-0   ISBN 978-3-662-12494-9 (eBook)
DOI 10.1007/978-3-662-12494-9

1. Metric spaces. 2. Geometry, Differential. I. Haefliger, André. II. Title. III. Series.
QA611.28.B75 1999 514'.32–dc21 99-38163 CIP

Mathematics Subject Classification (1991): 57C23, 20F32, 57Mxx, 53C70

ISSN 0072-7830

ISBN 978-3-642-08399-0

Originally published by Springer-Verlag Berlin Heidelberg New York in 1999
Cover design: MetaDesign plus GmbH, Berlin
Typesetting: Typeset by the authors and reformatted by Frank Herweg, Hirschberg-Leutershausen, using a Springer TEX macro-package
Printed on acid-free paper   SPIN: 10483755   41/3143/ko-5 4 3 2 1 0

For Julie and Minouche

Introduction

The purpose of this book is to describe the global properties of complete simply-connected spaces that are non-positively curved in the sense of A. D. Alexandrov and to examine the structure of groups that act properly on such spaces by isometries. Thus the central objects of study are metric spaces in which every pair of points can be joined by an arc isometric to a compact interval of the real line and in which every triangle satisfies the CAT(0) *inequality*. This inequality encapsulates the concept of non-positive curvature in Riemannian geometry and allows one to reflect the same concept faithfully in a much wider setting — that of geodesic metric spaces. Because the CAT(0) condition captures the essence of non-positive curvature so well, spaces that satisfy this condition display many of the elegant features inherent in the geometry of non-positively curved manifolds. There is therefore a great deal to be said about the global structure of CAT(0) spaces, and also about the structure of groups that act on them by isometries — such is the theme of this book.

The origins of our study lie in the fundamental work of A. D. Alexandrov[1]. He gave several equivalent definitions of what it means for a metric space to have curvature bounded above by a real number κ. Let us begin by explaining one of Alexandrov's definitions; this formulation has been given prominence by M. Gromov, who termed it the CAT(κ) inequality. (The initial A is in honour of Alexandrov, and the initials C and T are in honour of E. Cartan and A. Toponogov, each of whom made an important contribution to the understanding of curvature via inequalities for the distance function.)

Given a real number κ, let M_κ^2 denote the following space: if $\kappa < 0$ then M_κ^2 is real hyperbolic space $\mathbb{H}^2$ with the distance function scaled by a factor of $1/\sqrt{-\kappa}$; if $\kappa = 0$ then M_κ^2 is the Euclidean plane; if $\kappa > 0$ then M_κ^2 is the 2-sphere $\mathbb{S}^2$ with the metric scaled by a factor $1/\sqrt{\kappa}$. Alexandrov pointed out that one could define curvature bounds on a space by comparing triangles in that space to triangles in M_κ^2. A natural class of spaces in which to study triangles is the following. A metric space X is called a *geodesic space* if every pair of points $x, y \in X$ can be joined by a continuous path of length $d(x, y)$; the image of such a path is called a geodesic segment. In general there may be many geodesic segments joining x to y, but nevertheless it is convenient to use the notation $[x, y]$ for a choice of such a segment. A geodesic triangle Δ in X consists of three points $x, y, z \in X$ and three geodesic segments $[x, y]$, $[y, z]$, $[z, x]$. A *comparison triangle* for Δ in M_κ^2 is a geodesic triangle $\overline{\Delta}$ in M_κ^2 with vertices $\bar{x}, \bar{y}, \bar{z}$

[1] In 1957 Alexandrov wrote an article summarizing his ideas [Ale57]

such that $d(x, y) = d(\overline{x}, \overline{y})$, $d(y, z) = d(\overline{y}, \overline{z})$ and $d(z, x) = d(\overline{z}, \overline{x})$. (If $\kappa \leq 0$ then such a $\overline{\Delta}$ always exists; if $\kappa > 0$ then it exists provided the perimeter of Δ is less than $2\pi/\sqrt{\kappa}$; in both cases it is unique up to an isometry of M_κ^2.) The point $\overline{p} \in [\overline{x}, \overline{y}]$ is called *a comparison point* in $\overline{\Delta}$ for $p \in [x, y]$ if $d(x, p) = d(\overline{x}, \overline{p})$. Comparison points on $[y, z]$ and $[z, x]$ are defined similarly. A geodesic space X is said to satisfy the CAT(κ) inequality (more briefly, *X is a* CAT(κ) *space*) if, for all geodesic triangles Δ in X,

$$d(p, q) \leq d(\overline{p}, \overline{q})$$

for all comparison points $\overline{p}, \overline{q} \in \overline{\Delta} \subseteq M_\kappa^2$.

Alexandrov defines a metric space to be of *curvature* $\leq \kappa$ if each point of the space has a neighbourhood which, equipped with the induced metric, is a CAT(κ) space. He and the Russian school which he founded have made an extensive study of the local properties of such spaces. A complete Riemannian manifold has curvature $\leq \kappa$ in the above sense if and only if all of its sectional curvatures are $\leq \kappa$. The main point of making the above definition, though, is that there are many examples of spaces other than Riemannian manifolds whose curvature is bounded above. An interesting class of non-positively curved polyhedral complexes is provided by the buildings of Euclidean (or affine) type which arose in the work of Bruhat and Tits on algebraic groups. Many other examples will be described in the course of this book.

In recent years, CAT(-1) and CAT(0) spaces have come to play an important role both in the study of groups from a geometrical viewpoint and in the proofs of certain rigidity theorems in geometry. This is due in large part to the influence of Mikhael Gromov. Of particular importance are the lectures which Gromov gave in February 1981 at Collège de France in Paris. In these lectures (an excellent account of which was written by Viktor Schroeder [BGS95]) Gromov explained the main features of the global geometry of manifolds of non-positive curvature, essentially by basing his account on the CAT(0) inequality. In the present book we shall pursue this approach further in order to describe the global properties of CAT(0) spaces and the structure of groups which act on them by isometries. Two particular features of our treatment are that we give very detailed proofs of the basic theorems, and we describe many examples.

We have divided our book into three parts: Part I is an introduction to the geometry of geodesic spaces, in Part II the basic theory of spaces with upper curvature bounds is developed, and more specialized topics are covered in Part III. We shall now outline the contents of each part. Before doing so, we should emphasize that many of the chapters can be read independently, and we therefore suggest that if you are particularly interested in the material from a certain chapter, then you should turn directly to that chapter. (References are given in the text whenever material from earlier chapters[2] is needed.)

[2] We shall write (7.11) to direct readers to item 7.11 in the part of the book that they are reading, and (I.7.11) to direct readers to item 7.11 in Part I. The chapters in Part III are labelled by letters and subdivided into smaller sections, giving rise to references of the form (III.Γ.7.11).

In Part I we examine such basic concepts as distance (metric spaces), geodesics, the length of a curve, length (inner) metrics, and the notion of the (upper) angle between two geodesics issuing from the same point in a metric space. (This concept of angle, which is due to Alexandrov, plays an essential role throughout the book.) Part I also contains various examples of geodesic spaces. Of these, the most important are the model spaces M_κ^n, which we introduce in Chapter I.2 and study further in I.6. One can describe M_κ^n as the complete, simply connected, Riemannian n-manifold of constant sectional curvature κ. However, in keeping with the spirit of this book, we shall define M_κ^n directly as a metric space and deduce the desired properties of the space and its group of isometries directly from this definition.

We shall augment the supply of basic examples in Part I by describing several methods for constructing new examples of geodesic metric spaces out of more familiar ones: products, gluing, cones, spherical joins, quotients, induced path metrics and limits. Most of these constructions are due to Alexandrov and the Russian school. In Chapter I.7 we shall describe the general properties of geometric complexes, as established by Bridson in his thesis. And in the final chapter of Part I we shall turn our attention to groups: after gathering some basic facts about group actions, we shall describe some of the basic ideas in geometric group theory.

In Part II we set about our main task — exploring the geometry of CAT(κ) spaces. We shall give several different formulations of the CAT(κ) condition, all due to Alexandrov, and prove that they are equivalent. One quickly sees that CAT(κ) spaces enjoy significant properties. For example, one can see almost immediately that in a complete CAT(0) space angles exist in a strong sense, the distance function is convex, every bounded set has a unique circumcentre, one has orthogonal projections onto closed convex subsets, etc. Early in Part II we shall also examine how CAT(κ) spaces behave with regard to the basic constructions introduced in Chapter I.5.

Following these basic considerations, we turn our attention to a richer circle of ideas based on a key observation of Alexandrov: when considering a triangle Δ in a complete CAT(0) space X, if one gets any non-trivial equality in the CAT(0) condition, then Δ spans an isometrically embedded Euclidean triangle in X. This observation leads quickly to results concerning the existence of flat polygons and flat strips, and thence a product decomposition theorem.

Much of the force and elegance of the theory of non-positively curved spaces rests on the fact that there is a local-to-global theorem which allows one to use local information about the space to make deductions about the global geometry of its universal cover and about the structure of groups which act by isometries on the universal cover. More precisely, we have the following generalization of the Cartan-Hadamard theorem: for $\kappa \leq 0$, a complete simply-connected geodesic space satisfies the CAT(κ) inequality locally if and only if it satisfies the CAT(κ) inequality globally. (In Chapter II.4, following a proof of Alexander and Bishop, we shall actually prove a more general statement concerning metric spaces whose metrics are locally convex.)

A more concise account of much of the material presented in Chapters II.1-II.4 and II.8-II.9 of the present book can be found in the first two chapters of Ballmann's lecture notes [Ba95].

Even if one were ultimately interested only in CAT(0) spaces, there are aspects of the subject that force one to consider geodesic metric spaces satisfying the CAT(κ) inequality for arbitrary κ. An important link between CAT(0) spaces and CAT(1) spaces is provided by a theorem of Berestovskii, which shows that the Euclidean cone C_0Y over a geodesic space Y is a CAT(0) space if and only if Y is a CAT(1) space. (A similar statement holds with regard to the κ-cone $C_\kappa Y$, where κ is arbitrary.) This theorem is used in Chapter II.5 to establish the *link condition*, a necessary and sufficient condition (highlighted by Gromov) which translates questions concerning the existence of CAT(0) metrics on polyhedral complexes into questions concerning the structure of links of vertices. The importance of the link condition is that in many circumstances (particularly in dimension two) it provides a practical method for deciding if a given complex supports a metric of non-positive curvature. Thus we are able to construct interesting examples. Two-dimensional complexes are a particularly rich source of examples, partly because the link condition is easier to check than in higher dimensions, but also because the connections between group theory and geometry are closest in dimension two, and in dimension two any subcomplex of a non-positively curved complex is itself non-positively curved.

In Chapter II.6 we begin our study of groups that act by isometries on CAT(0) spaces. First we establish basic properties of individual isometries and groups of isometries. Individual isometries are divided into three classes according to the behaviour of their displacement functions. If the displacement function is constant then the isometry is called a Clifford translation. The Clifford translations of a CAT(0) space X form a pre-Hilbert space H, which is a generalization of the Euclidean de Rham factor in Riemannian geometry: if X is complete then there is an isometric splitting $X = X' \times H$. We also show that the group of isometries of a compact non-positively curved space is a topological group with finitely many connected components, the component of the identity being a torus.

In the early nineteen seventies, Gromoll-Wolf and Lawson-Yau proved several striking theorems concerning the structure of those groups that are the fundamental groups of compact non-positively curved Riemannian manifolds, including the Flat Torus Theorem, the Solvable Subgroup Theorem and the Splitting Theorem. In Chapters II.6 and II.7 we generalize these results to the case of groups that act properly and cocompactly by isometries on CAT(0) spaces. These generalizations have a variety of applications to group theory and topology.

In Chapters II.8 and II.9 we explore the geometry at infinity in CAT(0) spaces. Associated to any complete CAT(0) space one has a boundary at infinity ∂X, which can be constructed as the set of equivalence classes of geodesic rays in X, two rays being considered equivalent if their images are a bounded distance apart. There is a natural topology on $\overline{X} = X \cup \partial X$ called the cone topology. If X is complete and locally compact, $\overline{X}$ is compact. If X is a Riemannian manifold, $\overline{X}$ is homeomorphic to a closed ball, but for more general CAT(0) spaces the topology of ∂X can be rather complicated. An alternative construction of $\overline{X}$ is obtained by taking the closure of X in the Banach space C_*X of continuous functions on X modulo additive constants, where X is embedded in C_*X by the map that assigns to $x \in X$ the class of the function $y \mapsto d(x, y)$. In this description of $\overline{X}$ the points of ∂X emerge as classes of Busemann functions, and we are led to examine the geometry of horoballs in CAT(0) spaces.

There is a natural metric $\angle$ on the set ∂X: given $\xi, \mu \in \partial X$, one takes the supremum over all points $p \in X$ of the angle between the geodesics issuing from p in the classes ξ and μ. The topology on ∂X associated to this metric is in general weaker than the cone topology. (For instance if X is a CAT(−1) space, one gets the discrete topology.) We shall explain two significant facts concerning $\angle$: first, if X is a complete CAT(0) space then $(\partial X, \angle)$ is a CAT(1) space; secondly, the length metric associated to $\angle$, called the *Tits metric*, encodes the geometry of flat subspaces in X, in particular it determines how X can split as a product.

In Chapter III.H we shall revisit the study of boundaries in the context of Gromov's δ-hyperbolic spaces. In the context of CAT(0) spaces, the δ-hyperbolic condition is closely related to the idea of a visibility space, which was introduced in the context of smooth manifolds by Eberlein and O'Neill. Intuitively speaking, visibility spaces are "negatively curved on the large scale". In Chapter II.9 we shall see that if a proper CAT(0) space X admits a cocompact group of isometries, then X is a visibility space if and only if it does not contain an isometrically embedded copy of the Euclidean plane.

The main purpose of the remaining three chapters in Part II is to provide examples of CAT(0) spaces: in Chapter II.11 we describe various gluing techniques that allow one to build new examples out of more classical ones; in Chapter II.10 we describe elements of the geometry of symmetric spaces of non-compact type in terms of the metric approach to curvature developed in earlier chapters; and in Chapter II.12 we introduce simple complexes of groups as a forerunner to the general theory of complexes of groups developed in Chapter III.$\mathcal{C}$.

Complexes of groups were introduced by Haefliger to describe group actions on simply-connected polyhedral complexes in terms of suitable local data on the quotient. They are a natural generalization of the concept of a graph of groups, which is due to Bass and Serre. In order to work effectively with polyhedral complexes in this context, one needs a combinatorial description of them; the appropriate object to focus on is the partially ordered set of cells in the first barycentric subdivision of the complex, which provides the motivating example for objects that we call *scwols* (small categories without loops).

Associated to any action of a group on a scwol there is a complex of groups over the quotient scwol. If a complex of groups arises from such an action, it is said to be *developable*. In contrast to the one-dimensional case (graphs of groups), complexes of groups are not developable in general. However, if a complex of groups is non-positively curved, in a suitable sense, then it is developable.

The foundations of the theory of complexes of groups are laid out in Chapter III.$\mathcal{C}$. The developability theorem for non-positively curved complexes of groups is proved in Chapter III.$\mathcal{G}$, where it is treated in the more general context of groupoids of local isometries.

There are two other chapters in Part III. In the first, Chapter III.H, we describe elements of Gromov's theory of δ-hyperbolic metric spaces and discuss the relationship between non-positive curvature and isoperimetric equalities. In the second, Chapter III.Γ, we shall delve more deeply into the nature of groups that act properly and cocompactly by isometries on CAT(0) spaces. In particular, we shall analyse the algorithmic properties of such groups and explore the diverse nature of their subgroups. We shall also show that many theorems concerning groups of isometries of CAT(0) spaces can be extended to larger classes of groups — hyperbolic and semihyperbolic groups. The result is a substantial (but not comprehensive) account of the role which non-positive curvature plays in geometric group theory.

Having talked at some length about what this book contains, we should say a few words about what it does not contain. First we should point out that besides defining what it means for a metric space to have curvature bounded above, Alexandrov also defined what it means for a metric space to have curvature bounded below by a real number κ. (He did so essentially by imposing the reverse of the CAT(κ) inequality.) The theory of spaces with lower curvature bounds, particularly their local properties, has been developed extensively by Alexandrov and the Russian school, and such spaces play an important role in the study of collapsing for Riemannian manifolds. We shall not consider the theory of such spaces at all in this book, instead we refer the reader to the excellent survey article of Burago, Gromov and Perel'man [BGP92].

We should also make it clear that our treatment of the theory of non-positively curved spaces is by no means exhaustive; the study of such spaces continues to be a highly active field of research, encompassing many topics that we do not cover in this book. In particular, we do not discuss the conformal structure on the boundary of a CAT(-1) space, nor do we discuss the construction of Patterson measures at infinity, the geodesic flow in singular spaces of non-positive curvature, the theory of harmonic maps into CAT(0) spaces, rigidity theorems etc.

It is our intention that the present book should be able to serve as an introductory text. Although we shall arrive at non-trivial results, our lines of reasoning will be elementary, and we have written with the intention of making the material accessible to students whose background encompasses little more than a reasonable course in topology and an acquaintance with the basic concepts of group theory. Thus, for example, we expect the reader to understand what a manifold is and to be familiar with the definition of the fundamental group of a space, but a nodding acquaintance with the notion of a Riemannian metric will be quite sufficient for a complete understanding of this book. In any case, all such knowledge will be much less important than an enthusiasm for direct geometric arguments.

Acknowledgements: We thank the many colleagues whose comments helped to improve the content and exposition of the material presented in this book. In particular we thank Dick Bishop, Marc Burger, Mike Davis, Thomas Delzant, David Epstein, Pierre de la Harpe, Panos Papasoglu, Frédéric Paulin, John Roe, Ralph Strebel and Dani Wise.

We offer our heartfelt gratitude to Felice Ronga for his invaluable assistance in preparing this book for publication. We are particularly grateful for the long days that he spent transforming our rough sketches into the many figures that accompany the text and for his help in solving the many word processing problems we encountered.

We thank the Swiss National Science Foundation for its financial support and we thank the mathematics department at the University of Geneva for providing us with the facilities and equipment needed to prepare this book.

The first author thanks the Engineering and Physical Sciences Research Council of Great Britain for the Advanced Fellowship which currently supports his research. The National Science Foundation of America, the Alfred P. Sloane Foundation, and the Frank Buckley Foundation have also provided financial support for Bridson's work during the course of this project, and it is with pleasure that he takes this opportunity to thank each of them. Above all, he thanks his wife Julie Lynch Bridson for her constant love and support, and he thanks Minouche and André Haefliger for welcoming him so warmly into their home.

The second author expresses his deep gratitude to his wife Minouche for her unconditional support during his career, and for her dedication and interest during the preparation of this book.

Geneva, March 1999 MRB, AH

Table of Contents