

Susanne C. Brenner L. Ridgway Scott

The Mathematical Theory of Finite Element Methods

Second Edition

With 41 Illustrations



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