Undergraduate Texts in Mathematics

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Pierre Brémaud

An Introduction to Probabilistic Modeling

With 90 Illustrations

Springer

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To my father

Preface

The present textbook provides prerequisite material for courses in Physics, Electrical Engineering, Operations Research, and other fields of applied science where probabilistic models are used intensively. The emphasis has therefore been placed on *modeling* and *computation.*

There are two levels of modeling: abstract and concrete.

The abstract level is relative to the axiomatization of Probability and provides a general framework that features an archetype of all concrete models, where the basic objects (events, probability, random variables), the basic concepts (independence, expectation), and the basic rule (countable additivity of probability) are given in abstract form. This moderately small axiomatic equipment, establishing Probability as a mathematical theory, suffices to produce a theorem called the strong law of large numbers that says in particular that in tossing coins "the average number of heads tends to $\frac{1}{2}$ as the number of independent tosses tends to infinity, if the coin is fair." This result shows that the axioms of probability are consistent with empirical evidence. (From a mathematical point of view, this a posteriori check of the relevance of the axioms is not necessary, whereas from the point of view of the modeler, it is of course of paramount importance.)

In the present book, the abstract framework is immediately introduced and a number of examples showing how this framework relates to the daily concerns of physicists and engineers is provided. The strong law of large numbers where the abstract framework culminates is proved in the last chapter.

The other level of modeling consists of fitting a given situation into the conceptual framework of the axiomatic theory when it is believed that random phenomena occur. This is a difficult exercise at the beginning, and the art of modeling can be acquired only through examples. Supplementary readings—

entitled Illustrations— provide examples in which probabilistic models have been successfully developed.

They include, in particular, topics in *stochastic processes* and *statistics* as shown in the following list:

1. A Simple Model in Genetics: Mendel's Law and Hardy–Weinberg's Theorem
2. The Art of Counting: The Ballot Problem and the Reflection Principle
3. Bertrand's Paradox
4. An Introduction to Population Theory: Galton–Watson's Branching Process
5. Shannon's Source Coding Theorem: An Introduction to Information Theory
6. Buffon's Needle: A Problem in Random Geometry
7. An Introduction to Bayesian Decision Theory: Tests of Gaussian Hypotheses
8. A Statistical Procedure: The Chi-Square Test
9. Introduction to Signal Theory: Filtering.

The first chapter introduces the basic definitions and concepts of probability, independence, and cumulative distribution functions. It gives the elementary theory of conditioning (Bayes' formulas), and presents finite models, where computation of probability amounts to counting the elements of a given set. The second chapter is devoted to *discrete random variables* and to the generating functions of integer-valued random variables, whereas the third chapter treats the case of *random vectors admitting a probability density*. The last paragraph of the third chapter shows how Measure and Integration Theory can be useful to Probability Theory. It is of course just a brief summary of material far beyond the scope of an introduction to probability, emphasizing a useful technical tool: the Lebesgue convergence theorems. The fourth chapter treats two topics of special interest to engineers, operations researchers, and physicists: the *Gaussian vectors* and the *Poisson process*, which are the building blocks of a large number of probabilistic models. The treatment of Gaussian vectors is elementary but nevertheless contains the proof of the stability of the Gaussian character by extended linear transformations (linear transformations followed by passage to the limit in the quadratic mean). The Gaussian vectors and the Poisson process also constitute a source of examples of application of the formula of transformation of probability densities by smooth transformations of random vectors, which is given in the first paragraph and provides unity for this chapter. The last chapter treats the various concepts of *convergence*: in probability, almost sure, in distribution, and in the quadratic mean.

About 120 exercises with detailed solutions are presented in the main text to help the reader acquire computational skills and 28 additional exercises with outlines of solutions are given at the end of the book.

The material of the present textbook can be covered in a one-semester undergraduate course and the level can be adjusted simply by including or discarding portions of the last chapter, more technical, on convergences. The mathematical background consists of elementary calculus (series, Riemann integrals) and elementary linear algebra (matrices) as required of students in Physics and Engineering departments.

Gif-sur-Yvette, France PIERRE BRÉMAUD

Contents

Abbreviations and Notations

Abbreviations

a.s.	almost surely
c.d.f.	cumulative distribution function
c.f.	characteristic function
i.i.d.	independent and identically distributed
p.d.	probability density
q.m.	quadratic mean
r.v.	random variable

Notations

$\mathscr{B}(n, p)$	the binomial law of size n and parameter p (p. 47)
$\mathscr{G}(p)$	the geometric law of parameter p (p. 48)
$\mathscr{P}(\lambda)$	the Poisson law of mean λ (p. 49)
$\mathscr{M}(n, k, p_i)$	the multinomial law of size (n, k) and parameter $(p_1, \ldots, p_k)$ (p. 49)
$\mathscr{U}([a, b])$	the uniform law over $[a, b]$ (p. 86)
$\mathscr{E}(\lambda)$	the exponential law of parameter λ (p. 86)
$\mathscr{N}(m, \sigma^2)$	the Gaussian law of mean m and variance σ^2 (p. 87)
$\gamma(\alpha, \beta)$	the gamma law of parameters α and β (p. 88)
χ_n^2	the chi-square law with n degrees of freedom (p. 88)
$X \sim \cdots$	the random variable X is distributed according to ... (Example: "$X \sim \mathscr{E}(\lambda)$" means "$X$ is distributed according to the exponential law of parameter λ)
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^n$	the set of n-dimensional real vectors

$\mathscr{B}^n$	the Borelian sets of $\mathbb{R}^n$, that is: the smallest σ-field on $\mathbb{R}^n$ containing all the n-dimensional "rectangles"
$\mathbb{N}$	the set of non-negative integers
A'	transpose of the matrix A
u'	line vector, transpose of the column vector u