



NICOLAS BOURBAKI

ELEMENTS OF MATHEMATICS

General Topology

Chapters 1–4

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo

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ISBN-13: 978-3-540-64241-1 e-ISBN-13: 978-3-642-61701-0
DOI: 10.1007/978-3-642-61701-0

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Typesetting: Elsner & Behrens GmbH, Oftersheim
42/2202 - 5 4 3 2 1 0 - Printed on acid-free paper

ADVICE TO THE READER

1. This series of volumes, a list of which is given on pages 9 and 10, takes up mathematics at the beginning, and gives complete proofs. In principle, it requires no particular knowledge of mathematics on the reader's part, but only a certain familiarity with mathematical reasoning and a certain capacity for abstract thought. Nevertheless, it is directed especially to those who have a good knowledge of at least the content of the first year or two of a university mathematics course.

2. The method of exposition we have chosen is axiomatic and abstract, and normally proceeds from the general to the particular. This choice has been dictated by the main purpose of the treatise, which is to provide a solid foundation for the whole body of modern mathematics. For this it is indispensable to become familiar with a rather large number of very general ideas and principles. Moreover, the demands of proof impose a rigorously fixed order on the subject matter. It follows that the utility of certain considerations will not be immediately apparent to the reader unless he has already a fairly extended knowledge of mathematics; otherwise he must have the patience to suspend judgment until the occasion arises.

3. In order to mitigate this disadvantage we have frequently inserted examples in the text which refer to facts the reader may already know but which have not yet been discussed in the series. Such examples are always placed between two asterisks: * ... *. Most readers will undoubtedly find that these examples will help them to understand the text, and will prefer not to leave them out, even at a first reading. Their omission would of course have no disadvantage, from a purely logical point of view.

4. This series is divided into volumes (here called "Books"). The first six Books are numbered and, in general, every statement in the text


ADVICE TO THE READER

assumes as known only those results which have already been discussed in the preceding volumes. This rule holds good within each Book, but for convenience of exposition these Books are no longer arranged in a consecutive order. At the beginning of each of these Books (or of these chapters), the reader will find a precise indication of its logical relationship to the other Books and he will thus be able to satisfy himself of the absence of any vicious circle.

5. The logical framework of each chapter consists of the *definitions*, the *axioms*, and the *theorems* of the chapter. These are the parts that have mainly to be borne in mind for subsequent use. Less important results and those which can easily be deduced from the theorems are labelled as "propositions," "lemmas", "corollaries", "remarks", etc. Those which may be omitted at a first reading are printed in small type. A commentary on a particularly important theorem appears occasionally under the name of "scholium".

To avoid tedious repetitions it is sometimes convenient to introduce notations or abbreviations which are in force only within a certain chapter or a certain section of a chapter (for example, in a chapter which is concerned only with commutative rings, the word "ring" would always signify "commutative ring"). Such conventions are always explicitly mentioned, generally at the beginning of the chapter in which they occur.

6. Some passages in the text are designed to forewarn the reader against serious errors. These passages are signposted in the margin with the sign

 ("dangerous bend").

7. The Exercises are designed both to enable the reader to satisfy himself that he has digested the text and to bring to his notice results which have no place in the text but which are nonetheless of interest. The most difficult exercises bear the sign ¶.

8. In general, we have adhered to the commonly accepted terminology, except where there appeared to be good reasons for deviating from it.

9. We have made a particular effort to always use rigorously correct language, without sacrificing simplicity. As far as possible we have drawn attention in the text to *abuses of language*, without which any mathematical text runs the risk of pedantry, not to say unreadability.

10. Since in principle the text consists of the dogmatic exposition of a theory, it contains in general no references to the literature. Bibliographical references are gathered together in *Historical Notes*, usually

ADVICE TO THE READER

at the end of each chapter. These notes also contain indications, where appropriate, of the unsolved problems of the theory.

The bibliography which follows each historical note contains in general only those books and original memoirs which have been of the greatest importance in the evolution of the theory under discussion. It makes no sort of pretence to completeness; in particular, references which serve only to determine questions of priority are almost always omitted.

As to the exercises, we have not thought it worthwhile in general to indicate their origins, since they have been taken from many different sources (original papers, textbooks, collections of exercises).

II. References to a part of this series are given as follows :

- a) If reference is made to theorems, axioms, or definitions presented *in the same section*, they are quoted by their number.
- b) If they occur *in another section of the same chapter*, this section is also quoted in the reference.
- c) If they occur *in another chapter in the same Book*, the chapter and section are quoted.
- d) If they occur *in another Book*, this Book is first quoted by its title.

The *Summaries of Results* are quoted by the letter R: thus *Set Theory*, R signifies “*Summary of Results of the Theory of Sets*”.

CONTENTS

ADVICE TO THE READER	v
CONTENTS OF THE ELEMENTS OF MATHEMATICS SERIES	9
INTRODUCTION	11
CHAPTER I. <i>Topological Structures</i>	17
§ 1. Open sets, neighbourhoods, closed sets	17
1. Open sets	17
2. Neighbourhoods	18
3. Fundamental systems of neighbourhoods; bases of a topology	21
4. Closed sets	21
5. Locally finite families	22
6. Interior, closure, frontier of a set; dense sets	23
§ 2. Continuous functions	25
1. Continuous functions	25
2. Comparison of topologies	28
3. Initial topologies	30
4. Final topologies	32
5. Pasting together of topological spaces	34
§ 3. Subspaces, quotient spaces	35
1. Subspaces of a topological space	35
2. Continuity with respect to a subspace	37
3. Locally closed subspaces	38
4. Quotient spaces	39
5. Canonical decomposition of a continuous mapping ...	40
6. Quotient space of a subspace	42

CONTENTS

§ 4.	Product of topological spaces	43
	1. Product spaces	43
	2. Section of an open set; section of a closed set, projection of an open set. Partial continuity	46
	3. Closure in a product	47
	4. Inverse limits of topological spaces	48
§ 5.	Open mappings and closed mappings	50
	1. Open mappings and closed mappings	50
	2. Open equivalence relations and closed equivalence relations	52
	3. Properties peculiar to open mappings	54
	4. Properties peculiar to closed mappings	56
§ 6.	Filters	57
	1. Definition of a filter	57
	2. Comparison of filters	58
	3. Bases of a filter	59
	4. Ultrafilters	60
	5. Induced filter	61
	6. Direct image and inverse image of a filter base	62
	7. Product of filters	63
	8. Elementary filters	64
	9. Germs with respect to a filter	65
	10. Germs at a point	68
§ 7.	Limits	68
	1. Limit of a filter	68
	2. Cluster point of a filter base	69
	3. Limit point and cluster point of a function	70
	4. Limits and continuity	72
	5. Limits relative to a subspace	73
	6. Limits in product spaces and quotient spaces	74
§ 8.	Hausdorff spaces and regular spaces.....	75
	1. Hausdorff spaces	75
	2. Subspaces and products of Hausdorff spaces	77
	3. Hausdorff quotient spaces	78
	4. Regular spaces	80
	5. Extension by continuity; double limit	81
	6. Equivalence relations on a regular space	82
§ 9.	Compact spaces and locally compact spaces	83
	1. Quasi-compact spaces and compact spaces.....	83
	2. Regularity of a compact space	85

CONTENTS

3. Quasi-compact sets; compact sets; relatively compact sets	85
4. Image of a compact space under a continuous mapping	87
5. Product of compact spaces	88
6. Inverse limits of compact spaces	89
7. Locally compact spaces	90
8. Embedding of a locally compact space in a compact space	92
9. Locally compact σ -compact spaces	93
10. Paracompact spaces	94
§ 10. Proper mappings	97
1. Proper mappings	97
2. Characterization of proper mappings by compactness properties	101
3. Proper mappings into locally compact spaces	104
4. Quotient spaces of compact spaces and locally compact spaces	105
§ 11. Connectedness	107
1. Connected spaces and connected sets	107
2. Image of a connected set under a continuous mapping	109
3. Quotient spaces of a connected space	110
4. Product of connected spaces	110
5. Components	110
6. Locally connected spaces	112
7. Application : the Poincaré-Volterra theorem	113
Exercises for § 1	117
Exercises for § 2	119
Exercises for § 3	122
Exercises for § 4	125
Exercises for § 5	127
Exercises for § 6	129
Exercises for § 7	132
Exercises for § 8	133
Exercises for § 9	141
Exercises for § 10	150
Exercises for § 11	155
Historical Note	162
Bibliography	167

CONTENTS

CHAPTER II. <i>Uniform Structures</i>	169
§ 1. Uniform spaces	169
1. Definition of a uniform structure	169
2. Topology of a uniform space	171
§ 2. Uniformly continuous functions	174
1. Uniformly continuous functions	174
2. Comparison of uniformities	175
3. Initial uniformities	176
4. Inverse image of a uniformity; uniform subspaces	177
5. Least upper bound of a set of uniformities	178
6. Product of uniform spaces	179
7. Inverse limits of uniform spaces	180
§ 3. Complete spaces	181
1. Cauchy filters	181
2. Minimal Cauchy filters	183
3. Complete spaces	184
4. Subspaces of complete spaces	186
5. Products and inverse limits of complete spaces	186
6. Extension of uniformly continuous functions	190
7. The completion of a uniform space	191
8. The Hausdorff uniform space associated with a uniform space	195
9. Completion of subspaces and product spaces	197
§ 4. Relations between uniform spaces and compact spaces ...	198
1. Uniformity of compact spaces	198
2. Compactness of uniform spaces	200
3. Compact sets in a uniform space	203
4. Connected sets in a compact space	204
Exercises for § 1	206
Exercises for § 2	207
Exercises for § 3	208
Exercises for § 4	209
Historical Note	216
Bibliography	218
CHAPTER III : <i>Topological Groups</i>	219
§ 1. Topologies on groups	219
1. Topological groups	219

CONTENTS

2. Neighbourhoods of a point in a topological group	221
3. Isomorphisms and local isomorphisms	224
§ 2. Subgroups, quotient groups, homomorphisms, homogeneous spaces, product groups	225
1. Subgroups of a topological group	225
2. Components of a topological group	227
3. Dense subgroups	228
4. Spaces with operators	228
5. Homogeneous spaces	231
6. Quotient groups	232
7. Subgroups and quotient groups of a quotient group . .	233
8. Continuous homomorphisms and strict morphisms . . .	235
9. Products of topological groups	237
10. Semi-direct products	239
§ 3. Uniform structures on groups	242
1. The right and left uniformities on a topological group	242
2. Uniformities on subgroups, quotient groups and product groups	244
3. Complete groups	245
4. Completion of a topological group	246
5. Uniformity and completion of a commutative topological group	248
§ 4. Groups operating properly on a topological space; compact- ness in topological groups and spaces with operators	250
1. Groups operating properly on a topological space	250
2. Properties of groups operating properly	253
3. Groups operating freely on a topological space	254
4. Locally compact groups operating properly	255
5. Groups operating continuously on a locally compact space	257
6. Locally compact homogeneous spaces	259
§ 5. Infinite sums in commutative groups	261
1. Summable families in a commutative group	261
2. Cauchy's criterion	262
3. Partial sums; associativity	264
4. Summable families in a product of groups	266
5. Image of a summable family under a continuous homomorphism	267
6. Series	267
7. Commutatively convergent series	269

CONTENTS

§ 6. Topological groups with operators; topological rings, division rings and fields	271
1. Topological groups with operators	271
2. Topological direct sum of stable subgroups	272
3. Topological rings	274
4. Subrings; ideals; quotient rings; products of rings	276
5. Completion of a topological ring	276
6. Topological modules	278
7. Topological division rings and fields	281
8. Uniformities on a topological division ring	282
§ 7. Inverse limits of topological groups and rings.....	284
1. Inverse limits of algebraic structures	284
2. Inverse limits of topological groups and spaces with operators	286
3. Approximation of topological groups	289
4. Application to inverse limits	293
Exercises for § 1	296
Exercises for § 2	298
Exercises for § 3	306
Exercises for § 4	308
Exercises for § 5	314
Exercises for § 6	315
Exercises for § 7	324
Historical Note	327
Bibliography	328
 CHAPTER IV : <i>Real Numbers</i>	 329
§ 1. Definition of real numbers	329
1. The ordered group of rational numbers	329
2. The rational line	330
3. The real line and real numbers	331
4. Properties of intervals in \mathbf{R}	332
5. Length of an interval	333
6. Additive uniformity of \mathbf{R}	334
§ 2. Fundamental topological properties of the real line	334
1. Archimedes' axiom.....	334
2. Compact subsets of \mathbf{R}	335
3. Least upper bound of a subset of \mathbf{R}	335
4. Characterization of intervals.....	336

CONTENTS

5. Connected subsets of \mathbf{R}	336
6. Homeomorphisms of an interval onto an interval ...	338
§ 3. The field of real numbers	339
1. Multiplication in \mathbf{R}	339
2. The multiplicative group \mathbf{R}^*	340
3. n th roots	341
§ 4. The extended real line	342
1. Homeomorphism of open intervals of \mathbf{R}	342
2. The extended line	343
3. Addition and multiplication in $\overline{\mathbf{R}}$	345
§ 5. Real-valued functions	347
1. Real-valued functions	347
2. Real-valued functions defined on a filtered set	348
3. Limits on the right and on the left of a function of a real variable	349
4. Bounds of a real-valued function	350
5. Envelopes of a family of real-valued functions	352
6. Upper limit and lower limit of a real-valued function with respect to a filter	353
7. Algebraic operations on real-valued functions	356
§ 6. Continuous and semi-continuous real-valued functions ...	359
1. Continuous real-valued functions	359
2. Semi-continuous functions	360
§ 7. Infinite sums and products of real numbers	363
1. Families of positive finite numbers summable in \mathbf{R} ...	364
2. Families of finite numbers of arbitrary sign summable in \mathbf{R}	366
3. Product of two infinite sums	367
4. Families multipliable in \mathbf{R}^*	367
5. Summable families and multipliable families in \mathbf{R} ...	369
6. Infinite series and infinite products of real numbers ..	370
§ 8. Usual expansions of real numbers; the power of \mathbf{R}	373
1. Approximations to a real number	373
2. Expansions of real numbers relative to a base sequence	373
3. Definition of a real number by means of its expansion	374
4. Comparison of expansions	376

CONTENTS

5. Expansions to base a	376
6. The power of \mathbf{R}	377
Exercises for § 1	378
Exercises for § 2	381
Exercises for § 3	387
Exercises for § 4	388
Exercises for § 5	389
Exercises for § 6	393
Exercises for § 7	397
Exercises for § 8	401
Historical Note	406
Bibliography	417
Index of Notation (Chapters I-IV)	419
Index of Terminology (Chapters I-IV)	421