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Nilpotent Orbits, Primitive Ideals, and Characteristic Classes

A Geometric Perspective in Ring Theory

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