

Contents

<i>Preface</i>	<i>page xi</i>
<i>Terminology</i>	xv
1. Heights	1
1.1. Introduction	1
1.2. Absolute values	1
1.3. Finite-dimensional extensions	5
1.4. The product formula	9
1.5. Heights in projective and affine space	15
1.6. Heights of polynomials	21
1.7. Lower bounds for norms of products of polynomials	29
1.8. Bibliographical notes	33
2. Weil heights	34
2.1. Introduction	34
2.2. Local heights	35
2.3. Global heights	39
2.4. Weil heights	42
2.5. Explicit bounds for Weil heights	45
2.6. Bounded subsets	54
2.7. Metrized line bundles and local heights	57
2.8. Heights on Grassmannians	66
2.9. Siegel's lemma	72
2.10. Bibliographical notes	80
3. Linear tori	82
3.1. Introduction	82
3.2. Subgroups and lattices	82
3.3. Subvarieties and maximal subgroups	88
3.4. Bibliographical notes	92

vi Contents

4. Small points	93
4.1. Introduction	93
4.2. Zhang's theorem	93
4.3. The equidistribution theorem	101
4.4. Dobrowolski's theorem	107
4.5. Remarks on the Northcott property	117
4.6. Remarks on the Bogomolov property	120
4.7. Bibliographical notes	123
5. The unit equation	125
5.1. Introduction	125
5.2. The number of solutions of the unit equation	126
5.3. Applications	140
5.4. Effective methods	146
5.5. Bibliographical notes	149
6. Roth's theorem	150
6.1. Introduction	150
6.2. Roth's theorem	152
6.3. Preliminary lemmas	156
6.4. Proof of Roth's theorem	163
6.5. Further results	170
6.6. Bibliographical notes	174
7. The subspace theorem	176
7.1. Introduction	176
7.2. The subspace theorem	177
7.3. Applications	181
7.4. The generalized unit equation	186
7.5. Proof of the subspace theorem	197
7.6. Further results: the product theorem	226
7.7. The absolute subspace theorem and the Faltings–Wüstholz theorem	227
7.8. Bibliographical notes	230
8. Abelian varieties	231
8.1. Introduction	231
8.2. Group varieties	232
8.3. Elliptic curves	240
8.4. The Picard variety	246

8.5. The theorem of the square and the dual abelian variety	252
8.6. The theorem of the cube	257
8.7. The isogeny multiplication by n	263
8.8. Characterization of odd elements in the Picard group	265
8.9. Decomposition into simple abelian varieties	267
8.10. Curves and Jacobians	268
8.11. Bibliographical notes	282
9. Néron–Tate heights	283
9.1. Introduction	283
9.2. Néron–Tate heights	284
9.3. The associated bilinear form	289
9.4. Néron–Tate heights on Jacobians	294
9.5. The Néron symbol	301
9.6. Hilbert’s irreducibility theorem	314
9.7. Bibliographical notes	326
10. The Mordell–Weil theorem	328
10.1. Introduction	328
10.2. The weak Mordell–Weil theorem for elliptic curves	329
10.3. The Chevalley–Weil theorem	335
10.4. The weak Mordell–Weil theorem for abelian varieties	341
10.5. Kummer theory and Galois cohomology	344
10.6. The Mordell–Weil theorem	349
10.7. Bibliographical notes	351
11. Faltings’s theorem	352
11.1. Introduction	352
11.2. The Vojta divisor	356
11.3. Mumford’s method and an upper bound for the height	359
11.4. The local Eisenstein theorem	360
11.5. Power series, norms, and the local Eisenstein theorem	362
11.6. A lower bound for the height	371
11.7. Construction of a Vojta divisor of small height	376
11.8. Application of Roth’s lemma	381
11.9. Proof of Faltings’s theorem	387
11.10. Some further developments	391
11.11. Bibliographical notes	400

viii Contents

12. The <i>abc</i>-conjecture	401
12.1. Introduction	401
12.2. The <i>abc</i> -conjecture	402
12.3. Belyĭ's theorem	411
12.4. Examples	416
12.5. Equivalent conjectures	424
12.6. The generalized Fermat equation	435
12.7. Bibliographical notes	442
13. Nevanlinna theory	444
13.1. Introduction	444
13.2. Nevanlinna theory in one variable	444
13.3. Variations on a theme: the Ahlfors–Shimizu characteristic	457
13.4. Holomorphic curves in Nevanlinna theory	465
13.5. Bibliographical notes	477
14. The Vojta conjectures	479
14.1. Introduction	479
14.2. The Vojta dictionary	480
14.3. Vojta's conjectures	483
14.4. A general <i>abc</i> -conjecture	488
14.5. The <i>abc</i> -theorem for function fields	498
14.6. Bibliographical notes	513
Appendix A. Algebraic geometry	514
A.1. Introduction	514
A.2. Affine varieties	514
A.3. Topology and sheaves	518
A.4. Varieties	521
A.5. Vector bundles	525
A.6. Projective varieties	530
A.7. Smooth varieties	536
A.8. Divisors	544
A.9. Intersection theory of divisors	551
A.10. Cohomology of sheaves	563
A.11. Rational maps	574
A.12. Properties of morphisms	577
A.13. Curves and surfaces	581
A.14. Connexion to complex manifolds	583

Appendix B. Ramification	586
B.1. Discriminants	586
B.2. Unramified field extensions	591
B.3. Unramified morphisms	598
B.4. The ramification divisor	599
Appendix C. Geometry of numbers	602
C.1. Adeles	602
C.2. Minkowski's second theorem	608
C.3. Cube slicing	615
<i>References</i>	620
<i>Glossary of notation</i>	635
<i>Index</i>	643