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# Theory of Orbits

Volume 1:  
Integrable Systems and Non-perturbative Methods

With 71 Figures



Springer

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*Cover picture:* A wide-field view of the Carina region in the Southern Sky, kindly supplied by ESO (European Southern Observatory), with an insert from a miniature of the XII century *God architect of the cosmos*, miniature from “Bible moralisée”, Cod. 2554 f. 1v (Österreichische Nationalbibliothek, Vienna)

Cataloging-in-Publication Data applied for.

A catalog record for this book is available from the Library of Congress.

Bibliographic information published by Die Deutsche Bibliothek

Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>

**Corrected Third Printing 2004**

**ISSN 0941-7834**

**ISBN 978-3-642-08210-8**

**ISBN 978-3-662-03319-7 (eBook)**

**DOI 10.1007/978-3-662-03319-7**

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Originally published by Springer-Verlag Berlin Heidelberg New York in 1996

Softcover reprint of the hardcover 1st edition 1996

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Typesetting: Data conversion Frank Herweg,

Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN 10955321 55/3141/ba - 5 4 3 2 1 0

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