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Symmetries and Differential Equations

With 21 Illustrations

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Preface

In recent years there have been considerable developments in symmetry methods (group methods) for differential equations as evidenced by the number of research papers devoted to the subject. This is no doubt due to the inherent applicability of the methods to nonlinear differential equations. Symmetry methods for differential equations, originally developed by Sophus Lie, are highly algorithmic. They systematically unify and extend existing ad hoc techniques to construct explicit solutions for differential equations, most importantly for nonlinear differential equations. Often ingenious techniques for solving particular differential equations arise transparently from the group point of view, and thus it is somewhat surprising that symmetry methods are not more widely used.

A major portion of this book discusses work that has appeared since the publication of the book *Similarity Methods for Differential Equations,* Springer-Verlag, 1974, by G.W. Bluman and J.D. Cole. The present book includes a comprehensive treatment of Lie groups of transformations and thorough discussions of basic symmetry methods for solving ordinary and partial differential equations. No knowledge of group theory is assumed. Emphasis is placed on explicit computational algorithms to discover symmetries admitted by differential equations and to construct solutions resulting from symmetries.

This book should be particularly suitable for physicists, applied mathematicians, and engineers. Almost all of the examples are taken from physical and engineering problems including those concerned with heat conduction, wave propagation, and fluid flows. A preliminary version was used as lecture notes for a two-semester course taught by the first author at the University of British Columbia in 1987–88 to graduate and senior undergraduate students in applied mathematics and physics.

Chapters 1 through 4 encompass basic material. More specialized topics are covered in Chapters 5 through 7.

Chapter 1 introduces the basic ideas of group transformations and their connections with differential equations through a thorough treatment of dimensional analysis and generalizations of the well-known Buckingham Pi-theorem. This chapter should give the reader an intuitive grasp of the subject matter of the book in an elementary setting.

Chapter 2 develops the basic concepts of Lie groups of transformations and Lie algebras necessary in subsequent chapters. A Lie group of transformations is characterized in terms of its infinitesimal generators which form a Lie algebra.

Chapter 3 is concerned with ordinary differential equations. It is shown how group transformations are used to construct solutions and how to find group transformations leaving ordinary differential equations invariant. We present a reduction algorithm that reduces an nth order differential equation, admitting a solvable r-parameter Lie group of transformations, to an $(n-r)$th order differential equation plus r quadratures. We derive an algorithm to construct special solutions (invariant solutions) that are invariant under admitted Lie groups of transformations. For a first order differential equation such invariant solutions include separatrices and singular envelope solutions.

Chapter 4 is concerned with partial differential equations. It is shown how one finds group transformations leaving them invariant, how corresponding invariant solutions are constructed, and how group methods are applied to boundary value problems.

Chapter 5 discusses the connection between conservation laws and the invariance of Euler–Lagrange equations, arising from variational problems, under Lie groups of transformations. Various formulations of Noether's theorem are presented to construct such conservation laws. This leads to generalizing the concept of Lie groups of point transformations of earlier chapters to Lie–Bäcklund transformations that account for higher order conservation laws associated with partial differential equations that have solutions exhibiting soliton behavior. We present algorithms to construct recursion operators generating infinite sequences of Lie–Bäcklund symmetries.

In Chapter 6 it is shown how group transformations can be used to determine whether or not a given differential equation can be mapped invertibly to a target differential equation. Algorithms are given to construct such mappings when they exist. In particular, we give necessary and sufficient conditions for mapping a given nonlinear system of partial differential equations to a linear system of partial differential equations and for mapping a given linear partial differential equation with variable coefficients to a linear partial differential equation with constant coefficients.

In Chapter 7 the concept of Lie groups of transformations is generalized further to include nonlocal symmetries of differential equations. We present a systematic method for finding a special class of nonlocal symmetries that are realized as local symmetries of related auxiliary systems (potential symmetries). The introduction of potential symmetries significantly extends the applicability of group methods to both ordinary and partial differential equations. Together with the mapping algorithms developed in Chapter 6, the use of potential symmetries allows one to find systematically non-invertible mappings that transform nonlinear partial differential equations to linear partial differential equations.

Chapters 2 through 7 can be read independently of Chapter 1. The material in Chapter 2 is essential for all subsequent chapters but a reader only interested in scalar ordinary differential equations may omit Sections 2.3.3 to 2.3.5. Chapter 4 can be read independently of Chapter 3. A reader interested in conservation laws (Chapter 5) needs to know how to find Lie groups of transformations admitted by differential equations (Sections 3.2.3, 3.3.4, 4.2.3, 4.3.3). Chapter 6 can be read independently of Chapters 3 and 5.

Every topic is illustrated by examples. Almost all sections have many exercises. It is essential to do these exercises in order to obtain a working knowledge of the material. Each chapter ends with a Discussion section that puts its contents in perspective by summarizing major results, by referring to related works, and by introducing related material in subsequent chapters.

Within each section and subsection of a given chapter, definitions, theorems, lemmas, and corollaries are numbered separately as well as consecutively. For example, Definition 2.2.3-1 refers to the first definition and Theorem 2.2.3-1 to the first theorem in Section 2.2.3; Definition 1.4-1 refers to the first definition in Section 1.4. Exercises appear at the conclusion of a section; Exercise 1.3-4 refers to the fourth problem of Exercises 1.3.

We thank Greg Reid for helpful suggestions that improved Chapter 7, Alex Ma for his assistance, and Doug Jamison, Mei-Ling Fong, Sheila Hancock, Joanne Congo, Marilyn Lacate, Joan de Niverville, and Rita Sieber for their patience and care in typing various drafts of the manuscripts.

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