

Graduate Texts in Mathematics 125

*Editorial Board*

J.H. Ewing

F.W. Gehring

P.R. Halmos

# Graduate Texts in Mathematics

- 1 TAKEUTI/ZARING. Introduction to Axiomatic Set Theory. 2nd. ed.
- 2 OXTOBY. Measure and Category. 2nd ed.
- 3 SCHAEFFER. Topological Vector Spaces.
- 4 HILTON/STAMMBACH. A Course in Homological Algebra.
- 5 MAC LANE. Categories for the Working Mathematician.
- 6 HUGHES/PIPER. Projective Planes.
- 7 SERRE. A Course in Arithmetic.
- 8 TAKEUTI/ZARING. Axiomatic Set Theory.
- 9 HUMPHREYS. Introduction to Lie Algebras and Representation Theory.
- 10 COHEN. A Course in Simple Homotopy Theory.
- 11 CONWAY. Functions of One Complex Variable. 2nd ed.
- 12 BEALS. Advanced Mathematical Analysis.
- 13 ANDERSON/FULLER. Rings and Categories of Modules.
- 14 GOLUBITSKY/GUILEMIN. Stable Mappings and Their Singularities.
- 15 BERBERIAN. Lectures in Functional Analysis and Operator Theory.
- 16 WINTER. The Structure of Fields.
- 17 ROSENBLATT. Random Processes. 2nd ed.
- 18 HALMOS. Measure Theory.
- 19 HALMOS. A Hilbert Space Problem Book. 2nd ed., revised.
- 20 HUSEMOLLER. Fibre Bundles 2nd ed.
- 21 HUMPHREYS. Linear Algebraic Groups.
- 22 BARNES/MACK. An Algebraic Introduction to Mathematical Logic.
- 23 GREUB. Linear Algebra. 4th ed.
- 24 HOLMES. Geometric Functional Analysis and its Applications.
- 25 HEWITT/STROMBERG. Real and Abstract Analysis.
- 26 MANES. Algebraic Theories.
- 27 KELLEY. General Topology.
- 28 ZARISKI/SAMUEL. Commutative Algebra. Vol. I.
- 29 ZARISKI/SAMUEL. Commutative Algebra. Vol. II.
- 30 JACOBSON. Lectures in Abstract Algebra I. Basic Concepts.
- 31 JACOBSON. Lectures in Abstract Algebra II. Linear Algebra.
- 32 JACOBSON. Lectures in Abstract Algebra III. Theory of Fields and Galois Theory.
- 33 HIRSCH. Differential Topology.
- 34 SPITZER. Principles of Random Walk. 2nd ed.
- 35 WERMER. Banach Algebras and Several Complex Variables. 2nd ed.
- 36 KELLEY/NAMIOKA et al. Linear Topological Spaces.
- 37 MONK. Mathematical Logic.
- 38 GRAUERT/FRITZSCHE. Several Complex Variables.
- 39 ARVESON. An Invitation to  $C^*$ -Algebras.
- 40 KEMENY/SNELL/KNAPP. Denumerable Markov Chains. 2nd ed.
- 41 APOSTOL. Modular Functions and Dirichlet Series in Number Theory.
- 42 SERRE. Linear Representations of Finite Groups.
- 43 GILLMAN/JERISON. Rings of Continuous Functions.
- 44 KENDIG. Elementary Algebraic Geometry.
- 45 LOÈVE. Probability Theory I. 4th ed.
- 46 LOÈVE. Probability Theory II. 4th ed.
- 47 MOISE. Geometric Topology in Dimensions 2 and 3.

*continued after index*

Carlos A. Berenstein Roger Gay

# Complex Variables

## An Introduction

With 68 Illustrations



Springer-Verlag  
New York Berlin Heidelberg London  
Paris Tokyo Hong Kong Barcelona

Carlos A. Berenstein  
Department of Mathematics  
University of Maryland  
College Park, MD 20742  
USA

Roger Gay  
Department of Mathematics  
University of Bordeaux  
351 Cours de la Liberation  
33504 Talence, France

*Editorial Board*

J.H. Ewing  
Department of  
Mathematics  
Indiana University  
Bloomington, IN 47405  
USA

F.W. Gehring  
Department of  
Mathematics  
University of Michigan  
Ann Arbor, MI 48109  
USA

P.R. Halmos  
Department of  
Mathematics  
Santa Clara University  
Santa Clara, CA 95053  
USA

Library of Congress Cataloging-in-Publication Data

Berenstein, Carlos A.

Complex variables: an introduction / Carlos A. Berenstein, Roger  
Gay.

p. cm. — (Graduate texts in mathematics ; 125)

Includes bibliographical references (p. ) and index.

ISBN-13: 978-1-4612-7765-1

e-ISBN-13: 978-1-4612-3024-3

DOI: 10.1007/978-1-4612-3024-3

I. Functions of complex variables. I. Gay, R. (Roger), 1934–

II. Title. III. Series.

QA331.7.B46 1991

515'.9—dc20

90-39311

CIP

Printed on acid-free paper

© 1991 Springer-Verlag New York Inc.

Softcover reprint of the hardcover 1st edition 1991

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY 10010, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

Typeset by Asco Trade Typesetting Ltd., Hong Kong.

9 8 7 6 5 4 3 2 1

ISBN-13: 978-1-4612-7765-1

*To our families for their patience and encouragement*

*To Lars V. Ahlfors, with our admiration*

Nadie puede escribir un libro. Para  
Que un libro sea verdaderamente,  
Se requieren la aurora y el poniente,  
Siglos, armas y el mar que une y separa.

Jorge Luis Borges

# Preface

Textbooks, even excellent ones, are a reflection of their times. Form and content of books depend on what the students know already, what they are expected to learn, how the subject matter is regarded in relation to other divisions of mathematics, and even how fashionable the subject matter is. It is thus not surprising that we no longer use such masterpieces as Hurwitz and Courant's *Funktionentheorie* or Jordan's *Cours d'Analyse* in our courses.

The last two decades have seen a significant change in the techniques used in the theory of functions of one complex variable. The important role played by the inhomogeneous Cauchy–Riemann equation in the current research has led to the reunification, at least in their spirit, of complex analysis in one and in several variables. We say reunification since we think that Weierstrass, Poincaré, and others (in contrast to many of our students) did not consider them to be entirely separate subjects. Indeed, not only complex analysis in several variables, but also number theory, harmonic analysis, and other branches of mathematics, both pure and applied, have required a reconsideration of analytic continuation, ordinary differential equations in the complex domain, asymptotic analysis, iteration of holomorphic functions, and many other subjects from the classic theory of functions of one complex variable. This ongoing reconsideration led us to think that a textbook incorporating some of these new perspectives and techniques had to be written. In particular, we felt that introducing ideas from homological algebra, algebraic topology, sheaf theory, and the theory of distributions, together with the systematic use of the Cauchy–Riemann  $\bar{\partial}$ -operator, were essential to a complete understanding of the properties and applications of the holomorphic functions of one variable.

The idea that function theory can be integrated into other branches of mathematics is not unknown to our students. It is our experience that under-

graduates see many applications of complex analysis, such as the use of partial fractions, the Laplace transform, and the explicit computation of integrals and series which could not be done otherwise. Graduate students thus have a powerful motivation to understand the foundation of the theory of functions of one complex variable.

The present book evolved out of graduate courses given at the universities of Maryland and Bordeaux, where we have attempted to give the students a sense of the importance of new developments and the continuing vitality of the theory of functions. Because of the amount of material covered, we are presenting our work in two volumes.

We have tried to make this book self-contained and accessible to graduate students, while at the same time to reach quite far into the topics considered. For that reason we assume mainly knowledge that is found in the undergraduate curriculum, such as elementary linear algebra, calculus, and point set topology for the complex plane and the two-dimensional sphere  $S^2$ . Beyond this, we assume familiarity with metric spaces, the Hahn–Banach theorem, and the theory of integration as it can be found in many introductory texts of real analysis. Whenever we felt a subject was not universally known, we have given a short review of it.

Almost every section contains a large number of exercises of different levels of difficulty. Those that are not altogether elementary have been starred. Many starred exercises came from graduate qualifying examinations. Some exercises provide an insight into a subject that is explained in detail later in the text.

In the same vein, we have made each chapter, and sometimes each section, as independent as possible of the previous ones. If an argument was worth repeating, we did so. This is one of the reasons the formulas have not been numbered; when absolutely essential, they have been marked for ease of reference in their immediate neighborhood. There are some propositions and proofs that have also been starred, and the reader can safely skip them the first time around without loss of continuity. Finally, we have left for the second volume some subjects that require a somewhat better acquaintance with functional analysis.

Let us give a short overview of this volume. Some of the basic properties of holomorphic functions of one complex variable are really topological in nature. For instance, Cauchy's theorem and the theory of residues have a homotopy and a homology form. In the first chapter, we give a detailed description of differential forms (including a proof of the Stokes formula), homotopy theory, homology theory, and other parts of topology pertinent to the theory of functions in the complex plane. Later chapters introduce the reader to sheaf theory and its applications. We conclude Chapter 1 with the definition of holomorphic functions and with the properties of those functions that are immediate from the preceding topological considerations.

In the second chapter we study analytic properties of holomorphic functions, with emphasis on the notion of compact families. This permits an early

proof of the Riemann mapping theorem, and we explore some of its consequences and extensions. The class  $S$  of normalized univalent functions is introduced as an example of a compact family. A one-semester course in complex analysis could very well start in this chapter and refer the student back to selected topics as necessary.

In the third chapter we consider the solvability of the inhomogeneous Cauchy–Riemann equation. As a corollary we obtain a simple exposition of ideal theory and corresponding interpolation theorems in the algebra of holomorphic functions. We also study the boundary values of holomorphic functions in the sense of distributions, showing that every distribution on  $\mathbb{R}$  can be obtained as boundary value of a holomorphic function in  $\mathbb{C} \setminus \mathbb{R}$ . (An appendix to this chapter gives a short introduction to the concepts of distribution theory.) The Edge-of-the-Wedge theorem, an important generalization of the Schwarz reflection principle, is proven. These ideas lead directly to the theory of hyperfunctions to be considered in the second volume. We conclude this chapter with a totally new approach to the theory of residues.

In the fourth chapter we develop the theory of growth of subharmonic functions in such a way that Hadamard's infinite product expansion for entire functions of finite order is generalized to subharmonic functions. We give a proof due to Bell and Krantz of the fact that a biholomorphic mapping between smooth domains extends smoothly to the boundary. This is used to prove simply and rigorously properties of the Green function of a domain.

In order to develop fully the concept of analytic continuation, Chapter 5 has a short introduction to the theory of sheaves, covering spaces and Riemann surfaces. Among the applications of these ideas we give the index theorem for linear differential operators in the complex plane. This chapter also contains an introduction to the theory of Dirichlet series.

In the second volume the reader will find the application of the ideas and methods developed in the present volume to harmonic analysis, functional equations, and number theory. For instance, elliptic functions, mean-periodic functions, the corona theorem, the Bezout equation in spaces of entire functions, and the Leroy–Lindelöf theory of analytic continuation and its relation to functional equations and overconvergence of Dirichlet series.

This being a textbook, it is impossible to be entirely original, and we have benefited from the existence of many excellent monographs and even unpublished lecture notes, too many to give credit to all of them in every instance. The list of references contains their titles as well as those of a number of research articles relevant to the subjects we touched upon. In a few places we have also tried to steer the reader into further lines of study that were naturally related to the subject at hand, but that, due to the desire to keep this book within manageable limits, we were compelled to leave aside.

Almost everything that the reader will find in our book can be traced in one way or another to Ahlfors' *Complex Analysis*. When it appeared, it



changed entirely the way the subject was taught. Although we do not aspire to such achievement, our sincere hope is that we have not let him down.

Finally, we would like to thank Virginia Vargas for the excellent typing and her infinite patience. A number of our friends and students, among them, F. Colonna, D. Pascuas, A. Sebbar, A. Vidras, and A. Yger, have gladly played the role of guinea pigs, reading different portions of the manuscript and offering excellent advice. Our heartfelt thanks to all of them.

Carlos A. Berenstein  
Bethesda, Maryland

Roger Gay  
Saucats, La Brede

# Contents

Preface	vii
CHAPTER 1	
Topology of the Complex Plane and Holomorphic Functions	1
1.1. Some Linear Algebra and Differential Calculus	1
1.2. Differential Forms on an Open Subset $\Omega$ of $\mathbb{C}$	5
1.3. Partitions of Unity	15
1.4. Regular Boundaries	22
1.5. Integration of Differential Forms of Degree 2. The Stokes Formula	28
1.6. Homotopy. Fundamental Group	42
1.7. Integration of Closed 1-Forms Along Continuous Paths	56
1.8. Index of a Loop	66
1.9. Homology	71
1.10. Residues	84
1.11. Holomorphic Functions	91
CHAPTER 2	
Analytic Properties of Holomorphic Functions	98
2.1. Integral Representation Formulas	98
2.2. The Frechet Space $\mathcal{H}(\Omega)$	117
2.3. Holomorphic Maps	128
2.4. Isolated Singularities and Residues	138
2.5. Residues and the Computation of Definite Integrals	148
2.6. Other Applications of the Residue Theorem	165
2.7. The Area Theorem	178
2.8. Conformal Mappings	192
CHAPTER 3	
The $\bar{\partial}$ -Equation	213
3.1. Runge's Theorem	213

3.2. Mittag–Leffler’s Theorem	221
3.3. The Weierstrass Theorem	228
3.4. An Interpolation Theorem	235
3.5. Closed Ideals in $\mathcal{H}(\Omega)$	237
3.6. The Operator $\frac{\partial}{\partial \bar{z}}$ Acting on Distributions	245
3.7. Mergelyan’s Theorem	268
3.8. Short Survey of the Theory of Distributions. Their Relation to the Theory of Residues	278
CHAPTER 4	
Harmonic and Subharmonic Functions	299
4.1. Introduction	299
4.2. A Remark on the Theory of Integration	300
4.3. Harmonic Functions	304
4.4. Subharmonic Functions	318
4.5. Order and Type of Subharmonic Functions in $\mathbb{C}$	352
4.6. Integral Representations	366
4.7. Green Functions and Harmonic Measure	392
4.8. Smoothness up to the Boundary of Biholomorphic Mappings	413
4.9. Introduction to Potential Theory	433
CHAPTER 5	
Analytic Continuation and Singularities	480
5.1. Introduction	481
5.2. Elementary Study of Singularities and Dirichlet Series	482
5.3. A Brief Study of the Functions $\Gamma$ and $\zeta$	499
5.4. Covering Spaces	508
5.5. Riemann Surfaces	515
5.6. The Sheaf of Germs of Holomorphic Functions	523
5.7. Cocycles	534
5.8. Group Actions and Covering Spaces	541
5.9. Galois Coverings	544
5.10. The Exact Sequence of a Galois Covering	546
5.11. Universal Covering Space	550
5.12. Algebraic Functions, I	559
5.13. Algebraic Functions, II	565
5.14. The Periods of a Differential Form	572
5.15. Linear Differential Equations	583
5.16. The Index of Differential Operators	613
References	633
Notation and Selected Terminology	638
Index	646