Pi: A Source Book

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Preface

Our intention in this collection is to provide, largely through original writings, an extended account of pi from the dawn of mathematical time to the present. The story of pi reflects the most seminal, the most serious, and sometimes the most whimsical aspects of mathematics. A surprising amount of the most important mathematics and a significant number of the most important mathematicians have contributed to its unfolding—directly or otherwise.

Pi is one of the few mathematical concepts whose mention evokes a response of recognition and interest in those not concerned professionally with the subject. It has been a part of human culture and the educated imagination for more than twenty-five hundred years. The computation of pi is virtually the only topic from the most ancient stratum of mathematics that is still of serious interest to modern mathematical research. To pursue this topic as it developed throughout the millennia is to follow a thread through the history of mathematics that winds through geometry, analysis and special functions, numerical analysis, algebra, and number theory. It offers a subject that provides mathematicians with examples of many current mathematical techniques as well as a palpable sense of their historical development.

Why a Source Book?

Few books serve wider potential audiences than does a source book. To our knowledge, there is at present no easy access to the bulk of the material we have collected.

Both professional and amateur mathematicians, whether budding, blooming, or beginning to wilt, can find in it a source of instruction, study, and inspiration. Pi yields wonderful examples of how the best of our mathematical progenitors have struggled with a problem worthy of their mettle. One of the great attractions of the literature on pi is that it allows for the inclusion of very modern, yet still highly accessible, mathematics. Indeed, we have included several prize winning twentieth century expository papers, and at least half of the collected material dates from the last half of the twentieth century.

While this book is definitely a collection of literature on, and not a history of, pi, we anticipate that historians of mathematics will find the collection useful. As authors we believe that one legitimate way of exhibiting the history of a concept is in gathering a coherent collection of original and secondary sources, and then to let the documents largely tell their own stories when placed in an appropriate historical and intellectual context.

Equally, teachers at every level will find herein ample supplementary resources: for many purposes from material for special topic courses to preparatory information for seminars and colloquia and guidance for student projects.

What Is Included?

We have chosen to include roughly 70 representatives of the accumulated literature on pi. In the Contents each piece is accorded a very brief but hopefully illuminating description. This is followed by an Introduction in which we highlight some further issues raised by the collection. Finally, since the pre-Newtonian study of pi presents many more problems for the reader than does the material after the time of Huygens, we have included an Appendix *On the Early History of Pi.* We have also provided two other Appendices. *A Computational Chronology of Pi* offers a concise tabular accounting of computational records, and *Selected Formulae for Pi* presents a brief compendium of some of the most historically or computationally significant formulas for pi.

The pieces in the collection fall into three broad classes.

The core of the material is the accumulated mathematical research literature of four millennia. Although most of this comes from the last 150 years, there is much of interest from ancient Egypt, Greece, India, China, and medieval Islam. We trust that readers will appreciate the ingenuity of our earliest mathematicians in their valiant attempts to understand this number. The reader may well find this material as engrossing as the later work of Newton, Euler, or Ramanujan. Seminal papers by Lambert, Hermite, Lindemann, Hilbert and Mahler, to name but a few, are included in this category. Some of the more important papers on the number *e*, on zeta functions, and on Euler's constant have also been included as they are inextricably interwoven with the story of pi.

The second stratum of the literature comprises historical studies of pi, based on the above core sources, and of writings on the cultural meaning and significance of the number. Some of these are present here only in the bibliography such as Petr Beckmann's somewhat idiosyncratic monograph, *A History of Pi.* Other works on the subject are provided in extenso. These include Schepler's chronology of pi, some of Eves's anecdotes about the history of the number, and Engels' conjecture about how the ancient Egyptians may have computed pi.

Finally, the third level comprises the treatments of pi that are fanciful, satirical or whimsical, or just wrongheaded. Although these abound, we have exercised considerable restraint in this category and have included only a few representative pieces such as Keith's elaborate mnemonic for the digits of pi based on the poem "The Raven," a recent offering by Umberto Eco, and the notorious 1897 attempt by the state of Indiana[1] to legislate the value of pi.

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September 6, 1996

[1]Oddly enough, the third page of this bill is apparently missing from the Indiana State Library and thus may now exist only in facsimile!

Some Points of Entry

For the reader looking for accessible points of introduction to the collection we make the following suggestions:

- As a general introduction:

- As an introduction to irrationality and transcendence:

- As an introduction to elliptic integrals and related subjects:

- As an introduction to the computational issues:

Acknowledgments

We would like to thank, first of all, the publishers and authors who graciously granted permission to reproduce the works contained in this volume. Our principal debt, however, is to our technical editor, Chiara Veronesi, whose hard work and intelligent grasp of what needed to be done made the timely appearance of this book possible. We also wish to thank the publisher, Springer-Verlag, for its enthusiastic response to this project, as well as Ina Lindemann, our editor at Springer-Verlag, who saw the project through the press. Thanks, also, are due to David Fowler, who supplied copies of the Latin material contained herein from the work of John Wallis, as well as to David Bailey, Greg Fee and Yasumasa Kanada for helpful conversations about the project. Finally, we wish to thank the *Social Sciences Research Council of Canada* Small Grants Committee at Simon Fraser University for funding (Grant No. 410-86-0805) part of the cost of preparing this volume.

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Contents

Introduction

As indicated in the Preface, the literature on pi naturally separates into three components (primary research, history, and exegesis). It is equally profitable to consider three periods (before Newton, Newton to Hilbert and the Twentieth Century) and two major stories (pi's transcendence and pi's computation). With respect to computation, it is also instructive to consider the three significant methods which have been used: pre-calculus (Archimedes' method of exhaustion), calculus (Machin-like arctangent formulae), and elliptic and modular function methods (the Gaussian arithmetic–geometric mean and the series of Ramanujan type).

In the following introduction to the papers from the three periods we have resisted the temptation to turn our *Source Book* into a "History of Pi and the Methods for Computing it." Accordingly, we have made no attempt to give detailed accounts of any of the papers selected, even when the language or style might seem to render such accounts desirable. Instead, we urge the reader seeking an account of 'what's going on' to either consult a reliable general history of mathematics, such as that of C. Boyer (in its most recent up-date by U. Merzbach) or V. Katz, or P. Beckmann's more specialized and personalized history of pi.

The Pre-Newtonian Period (Papers [1] to [15])

The primary sources for this period are, not surprisingly, more problematic than those of later periods, and for this reason we have included an additional appendix on this material. Our selections visit Egyptian, Greek, Chinese, and Medieval Arabo-European traditions. We commence with an excerpt from the Rhind Mathematical Papyrus from the Middle Kingdom in Egypt, circa 1650 B.C., representing some of what the ancient Egyptians knew about mathematics around 1800 B.C. By far the most significant ancient work – that of Archimedes of Syracuse (277–212 B.C.), which survives under the title *On the Measurement of the Circle* follows. It is hard to overemphasize how this work dominated the subject prior to the advent of the calculus.

We continue with a study of Liu Hui's third century A.D. commentary on the Chinese classic *Nine Chapters in the Mathematical Art* and of the lost work of the fifth century astronomer Zu Chongzhi. Marshall Clagett's translation of *Verba Filiorum*, the Latin version of the 9th century Arabic *Book of Knowledge of the Measurement of Plane and Spherical Figures* completes our first millenium extracts.

The next selection jumps forward 500 years and discusses the tombstone of Ludolph van Ceulen which recorded the culminating computation of pi by purely Archimedian techniques to 35 places as performed by Ludolph, using 2^{62}-gons, before 1615. We complete this period with excerpts from three great transitional thinkers: François Viète (1540–1603) whose work greatly influenced that of Fermat; John Wallis (1616–1703), to whom Newton indicated great indebtedness; and the Dutch polymath Christian Huygens (1629–1695), who correctly formalized Willebrord Snell's acceleration of Archimedes' method and was thus able to recapture Van Ceulen's computation with only 2^{30}-gons. In a part of this work, not reproduced here, Huygens vigorously attacks the validity of Gregory's argument for the transcendence of pi.

From Newton to Hilbert (Papers [16] to [24])

These comprise many of the most significant papers on pi. After visiting Newton's contribution we record a discussion of the arctangent series for pi variously credited to the Scot James Gregory, the German Leibniz, and to the earlier Indian Mādhava. In this period we move from the initial investigations of irrationality, by Euler and Lambert, to one of the landmarks of nineteenth century mathematics, the proof of the transcendence of pi.

The first paper is a selection from Euler and it demonstrates Euler's almost unparalleled—save for Ramanujan—ability to formally manipulate series, particularly series for pi. It is followed by an excerpt from Lambert and a discussion by Struik of Lambert's proof of the irrationality of pi, which is generally credited as the first proof of its irrationality. Euler had previously proved the irrationality of *e*. Lambert's proof of the irrationality of pi is based on a complicated continued fraction expansion. Much simpler proofs are to be found in [33], [48].

There is a selection from Shank's self-financed publication that records his hand calculation of 607 digits of pi. (It is in fact correct only to 527 places, but this went unnoticed for almost a century.) The selection is included to illustrate the excesses that this side of the story has evoked. With a modern understanding of accelerating calculations this computation, even done by hand, could be considerably simplified. Neither Shanks's obsession with the computation of digits nor his error are in any way unique. Some of this is further discussed in [64].

The next paper is Hermite's 1873 proof of the transcendence of *e*. It is followed by Lindemann's 1882 proof of the transcendence of pi. These are, arguably, the most important papers in the collection. The proof of the transcendence of pi laid to rest the possibility of "squaring the circle," a problem that had been explicit since the late 5th c. B.C. Hermite's seminal paper on *e* in many ways anticipates Lindemann, and it is perhaps surprising that Hermite did not himself prove the transcendence of pi. The themes of Hermite's paper are explored and expanded in a number of later papers in this volume. See in particular Mahler [42]. The last two papers offer simplified proofs of the transcendence. One is due to Weierstrass in 1885 and the other to Hilbert in 1893. Hilbert's elegant proof is still probably the simplest proof we have.

The Twentieth Century (Papers [26] to [70])

The remaining forty-five papers are equally split between analytic and computational selections, with an interweaving of more diversionary selections.

On the analytic side we commence with the work of Ramanujan. His 1914 paper, [29], presents an extraordinary set of approximations to pi via "singular values" of elliptic integrals. The first half of this paper was well studied by Watson and others in the 1920s and 1930s, while the second half, which presents marvelous series for pi, was decoded and applied only more than 50 years later. (See [61], [62], [63].) Other highlights include: Watson's engaging and readable account of the early development of elliptic functions, [30]; several very influential papers by Kurt Mahler; Fields Medalist Alan Baker's 1964 paper on "algebraic independence of logarithms," [40]; and two papers on the irrationality of $\zeta(3)$ ([48], [49]) which was established only in 1976.

The computational selections include a report on the early computer calculation of pi—to 2037 places on ENIAC in 1949 by Reitwiesner, Metropolis and Von Neumann [34] and the 1961 computation of pi to 100,000 places by Shanks and Wrench [38], both by arctangent methods. Another highlight is the independent 1976 discovery of arithmetic-geometric mean methods for the computation of pi by Salamin and by Brent ([46], [47], see also [57]). Recent supercomputational applications of these and related methods by Kanada, by Bailey, and by the Chudnovsky brothers are included (see [60] to [64]). As of going to press, these scientists have now pushed the record for computation of pi beyond 17 billion digits. (See Appendix II.) One of the final papers in the volume, [70], describes a method of computing individual binary digits of pi and similar polylogarithmic constants and records the 1995 computation of the ten billionth hexadecimal digit of pi.