Graduate Texts in Mathematics 100

Graduate Texts in Mathematics

1 TAKEUTI/ZARING. Introduction to Axiomatic Set Theory. 2nd ed.
2 OXTOBY. Measure and Category. 2nd ed.
3 SCHAEFFER. Topological Vector Spaces.
4 HILTON/STAMMBACH. A Course in Homological Algebra.
5 MACLANE. Categories for the Working Mathematician.
6 HUGHES/PIPER. Projective Planes.
7 SERRE. A Course in Arithmetic.
8 TAKEUTI/ZARING. Axiometic Set Theory.
9 HUMPHREYS. Introduction to Lie Algebras and Representation Theory.
10 COHEN. A Course in Simple Homotopy Theory.
11 CONWAY. Functions of One Complex Variable. 2nd ed.
12 BEALS. Advanced Mathematical Analysis.
13 ANDERSON/FULLER. Rings and Categories of Modules.
14 GOLUBITSKY/GUILLEMIN. Stable Mappings and Their Singularities.
15 BERBERIAN. Lectures in Functional Analysis and Operator Theory.
16 WINTER. The Structure of Fields.
17 ROSENBLATT. Random Processes. 2nd ed.
18 HALMOS. Measure Theory.
19 HALMOS. A Hilbert Space Problem Book. 2nd ed., revised.
20 HUSEMOLLER. Fibre Bundles. 2nd ed.
21 HUMPHREYS. Linear Algebraic Groups.
22 BARNES/MACK. An Algebraic Introduction to Mathematical Logic.
23 GREUB. Linear Algebra. 4th ed.
24 HOLMES. Geometric Functional Analysis and its Applications.
25 HEWITT/STROMBERG. Real and Abstract Analysis.
26 MANES. Algebraic Theories.
27 KELLEY. General Topology.
28 ZARISKI/SAMUEL. Commutative Algebra. Vol. I.
29 ZARISKI/SAMUEL. Commutative Algebra. Vol. II.
30 JACOBSON. Lectures in Abstract Algebra I: Basic Concepts.
31 JACOBSON. Lectures in Abstract Algebra II: Linear Algebra.
32 JACOBSON. Lectures in Abstract Algebra III: Theory of Fields and Galois Theory.
33 HIRSCH. Differential Topology.
34 SPITZER. Principles of Random Walk. 2nd ed.
35 WERMER. Banach Algebras and Several Complex Variables. 2nd ed.
36 KELLEY/NAMIOKA et al. Linear Topological Spaces.
37 MONK. Mathematical Logic.
38 GRAUERT/FRITZSCHE. Several Complex Variables.
39 ARVESON. An Invitation to C^*-Algebras.
40 KEMENY/SNELL/KNAPP. Denumerable Markov Chains. 2nd ed.
41 APOSTOL. Modular Functions and Dirichlet Series in Number Theory.
42 SERRE. Linear Representations of Finite Groups.
43 GILLMAN/JERISON. Rings of Continuous Functions.
44 KENDIG. Elementary Algebraic Geometry.
45 LOÈVE. Probability Theory I. 4th ed.
46 LOÈVE. Probability Theory II. 4th ed.
47 MOISE. Geometric Topology in Dimensions 2 and 3.

continued after Index

Christian Berg
Jens Peter Reus Christensen
Paul Ressel

Harmonic Analysis on Semigroups

Theory of Positive Definite and Related Functions

Springer Science+Business Media, LLC

Christian Berg
Jens Peter Reus Christensen
Matematisk Institut
Københavns Universitet
Universitetsparken 5
DK-2100 København Ø
Denmark

Paul Ressel
Mathematisch-Geographische Fakultät
Katholische Universität Eichstätt
Residenzplatz 12
D-8078 Eichstätt
Federal Republic of Germany

AMS Classification (1980) *Primary*: 43-02, 43A35
Secondary: 20M14, 28C15, 43A05, 44A10, 44A60, 46A55, 52A07, 60E15

Library of Congress Cataloging in Publication Data
Berg, Christian
Harmonic analysis on semigroups.
(Graduate texts in mathematics; 100)
Bibliography: p.
Includes index.
1. Harmonic analysis. 2. Semigroups. I. Christensen, Jens Peter Reus. II. Ressel, Paul. III. Title. IV. Series.
QA403.B39 1984 515′.2433 83-20122

With 3 Illustrations.

Originally published by Springer-Verlag Berlin Heidelberg New York Tokyo in 1984
Softcover reprint of the hardcover 1st edition 1984

Typeset by Composition House Ltd., Salisbury, England.

9 8 7 6 5 4 3 2 1

ISBN 978-1-4612-7017-1 ISBN 978-1-4612-1128-0 (eBook)
DOI 10.1007/978-1-4612-1128-0

Preface

The Fourier transform and the Laplace transform of a positive measure share, together with its moment sequence, a positive definiteness property which under certain regularity assumptions is characteristic for such expressions. This is formulated in exact terms in the famous theorems of Bochner, Bernstein–Widder and Hamburger. All three theorems can be viewed as special cases of a general theorem about functions φ on abelian semigroups with involution $(S, +, *)$ which are positive definite in the sense that the matrix $(\varphi(s_j^* + s_k))$ is positive definite for all finite choices of elements $s_1, \ldots, s_n$ from S. The three basic results mentioned above correspond to $(\mathbb{R}, +, x^* = -x)$, $([0, \infty[, +, x^* = x)$ and $(\mathbb{N}_0, +, n^* = n)$.

The purpose of this book is to provide a treatment of these positive definite functions on abelian semigroups with involution. In doing so we also discuss related topics such as negative definite functions, completely monotone functions and Hoeffding-type inequalities. We view these subjects as important ingredients of harmonic analysis on semigroups. It has been our aim, simultaneously, to write a book which can serve as a textbook for an advanced graduate course, because we feel that the notion of positive definiteness is an important and basic notion which occurs in mathematics as often as the notion of a Hilbert space. The already mentioned Laplace and Fourier transformations, as well as the generating functions for integer-valued random variables, belong to the most important analytical tools in probability theory and its applications. Only recently it turned out that positive (resp. negative) definite functions allow a probabilistic characterization in terms of so-called Hoeffding-type inequalities.

As prerequisites for the reading of this book we assume the reader to be familiar with the fundamental principles of algebra, analysis and probability, including the basic notions from vector spaces, general topology and abstract measure theory and integration. On this basis we have included Chapter 1 about locally convex topological vector spaces with the main objective of proving the Hahn–Banach theorem in different versions which will be used later, in particular, in proving the Krein–Milman theorem. We also present a short introduction to the idea of integral representations in compact convex sets, mainly without proofs because the only version of Choquet's theorem which we use later is derived directly from the Krein–Milman theorem. For later use, however, we need an integration theory for measures on Hausdorff spaces, which are not necessarily locally compact. Chapter 2 contains a treatment of Radon measures, which are inner regular with respect to the family of compact sets on which they are assumed finite. The existence of Radon product measures is based on a general theorem about Radon bimeasures on a product of two Hausdorff spaces being induced by a Radon measure on the product space. Topics like the Riesz representation theorem, adapted spaces, and weak and vague convergence of measures are likewise treated.

Many results on positive and negative definite functions are not really dependent on the semigroup structure and are, in fact, true for general positive and negative definite matrices and kernels, and such results are placed in Chapter 3.

Chapters 4–8 contain the harmonic analysis on semigroups as well as a study of many concrete examples of semigroups. We will not go into detail with the content here but refer to the Contents for a quick survey. Much work is centered around the representation of positive definite functions on an abelian semigroup $(S, +, *)$ with involution as an integral of semicharacters with respect to a positive measure. It should be emphasized that most of the theory is developed without topology on the semigroup S. The reason for this is simply that a satisfactory general representation theorem for continuous positive definite functions on topological semigroups does not seem to be known. There is, of course, the classical theory of harmonic analysis on locally compact abelian groups, but we have decided not to include this in the exposition in order to keep it within reasonable bounds and because it can be found in many books.

As described we have tried to make the book essentially self-contained. However, we have broken this principle in a few places in order to obtain special results, but have never done it if the results were essential for later development. Most of the exercises should be easy to solve, a few are more involved and sometimes require consultations in the literature referred to. At the end of each chapter is a section called Notes and Remarks. Our aim has not been to write an encyclopedia but we hope that the historical comments are fair.

Within each chapter sections, propositions, lemmas, definitions, etc. are numbered consecutively as 1.1, 1.2, 1.3, . . . in §1, as 2.1, 2.2, 2.3, . . . in §2, and so on. When making a reference to another chapter we always add the number of that chapter, e.g. 3.1.1.

We have been fascinated by the present subject since our 1976 paper and have lectured on it on various occasions. Research projects in connection with the material presented have been supported by the Danish Natural Science Research Council, die Thyssen Stiftung, den Deutschen Akademischen Austauschdienst, det Danske Undervisningsministerium, as well as our home universities. Thanks are due to Flemming Topsøe for his advice on Chapter 2. We had the good fortune to have Bettina Mann type the manuscript and thank her for the superb typing.

March 1984

CHRISTIAN BERG
JENS PETER REUS CHRISTENSEN
PAUL RESSEL

Contents